Ma 147, Hamiltonian Dynamics

Spring 2007

MIDTERM EXAM

Due Wednesday May 2, 2:30 pm, 2007.

You may use your notes for this course, your solutions to course homework, and the textbooks. No collaboration or a direct assistance of other people.

All problems are weighted equally. No time limits.

1) Draw the phase portrait of the hamiltonian vector field on $\mathbb{R}^2$ for the Hamiltonian

a) $H(p,q) = \cos p + \cos q$

b) $H(p,q) = \cos p + \cos\left(p + \frac{\sqrt{3}}{2}q\right) + \cos\left(p - \frac{\sqrt{3}}{2}q\right)$

2) Consider a central force field on $\mathbb{R}^2$, $F(\bar{r}) = -f(|\bar{r}|)e_{\bar{r}}$, $f(r) = \frac{k^2}{r^5}$. Show that a particle in this force field may describe a circle passing through the origin.

3) Consider a polar coordinate diffeomorphism ρ from the upper half of the cylinder $\mathbb{R} \times S^1$ onto $\mathbb{R}^2 \backslash \{0\}$, defined by $(r,\theta) \to (r\cos\theta, r\sin\theta)$. Show that relative to the volume $d(\frac{r^2}{2}) \wedge d\theta$ on $\mathbb{R} \times S^1$ and to the standard one on $\mathbb{R}^2$ the map ρ is symplectic.

4) Assume that V is a smooth submanifold of a manifold M, and on M two smooth vector fields X and Y tangent to V are given. Prove that the Poisson bracket $[X,Y]$ is also a vector field tangent to V.

5) Let A be a $2n \times 2n$ constant Hamiltonian matrix. Consider a linear Hamiltonian system

$$\dot{z} = Az = J\frac{\partial H}{\partial z} \tag{1}$$

This system is called *stable* if all solutions are bounded for all $t \in \mathbb{R}$. It is called *parametrically stable* if it and all sufficiently small constant linear Hamiltonian perturbations of it are stable. Prove the following:

a) The system (1) is stable if and only if all the eigenvalues of A are pure imaginary, and A is diagonalizable (over the complex numbers).

b) If the Hamiltonian H is positive (or negative) definite, then the system (1) is parametrically stable.

c) If the system (1) is parametrically stable, then zero is not an eigenvalue of the matrix A.