Classical Analysis – Math 108b

Homework #7
due 11 am Monday, February 28, 2005.

1. Prove that, for any $f \in C[0,1]$,

$$\left| \int_0^1 f(t) \sin \pi t dt \right| \leq \frac{1}{\sqrt{2}} \left\{ \int_0^1 |f(t)|^2 dt \right\}^{1/2},$$

and describe the functions f for which equality holds.

2. Find an inner product on the space of polynomials with real coefficients such that the corresponding norm is given by

$$\|f\| = \left\{ \int_{-1}^1 |x||f(x)|^2 + 3|f'(x)|^2 dx \right\}^{1/2}.$$

Prove that, for any polynomial f,

$$\left| \int_{-1}^1 |x|^3 f(x) + 6x f'(x) dx \right| \leq \frac{5}{\sqrt{3}} \left\{ \int_{-1}^1 |x||f(x)|^2 + 3|f'(x)|^2 dx \right\}^{1/2}.$$

3. Prove that there exists an orthonormal basis in $L_2[0,1]$ which contains only polynomials.

4. Let $0 < c < 1$. Show that the linear functional T on $C[0,1]$ defined by $T(x) = x(c), \quad x \in C[0,1]$, is continuous with respect to the supremum norm, but not with respect to the $L_2[0,1]$ norm (restricted to $C[0,1]$).

5. Let $l^2_{\mathbb{Z}}$ denote the Hilbert space of square-summable sequences $\{x_n\}_{-\infty}^{\infty}$ of complex numbers indexed by $\mathbb{Z}$, with componentwise algebraic operations and inner product

$$(x, y) = \sum_{-\infty}^{\infty} x_n \bar{y}_n.$$

Write down an explicit isomorphism between l^2 and $l^2_{\mathbb{Z}}$.

References.

1. N.Young, "An introduction to Hilbert space", Cambridge University Press, 1988.
2. W.Rudin, "Real and Complex Analysis", any edition.