Classical Analysis – Math 108b

Homework #6
due 11 am Tuesday, February 22, 2005.

1. Let $\{g_n\}$ be a sequence of nonnegative integrable functions possessing the properties

$$i)\ \lim_{n\to\infty} g_n(x) = g(x) \text{ exists at a.e. } x \in \mathbb{R}, \quad ii)\ \lim_{n\to\infty} \int_{\mathbb{R}} g_n = \int_{\mathbb{R}} g.$$

Suppose another sequence of functions $\{f_n\}$ satisfies $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere and also $|f_n(x)| \le g_n(x)$ for each n.

Prove that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.

2. Let $f \in L_1(\mathbb{R})$ and define $\varphi(x) = \sum_{n=1}^{\infty} f(2^n x + \frac{1}{n})$. Show that φ is integrable and that $\int f = \int \varphi$.

3. Recall that the convolution of two functions f, g is defined as $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. Suppose that $f \in L_1(\mathbb{R})$ and $g \in L_\infty(\mathbb{R})$. Show that $f * g$ is a bounded and continuous function.

4. Let $p, q, r > 1$ be any choice of real numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Prove the following extension of Hölder's Inequality:

$$\int_{\mathbb{R}} |f(x)g(x)h(x)| \le \|f\|_{L_p} \|g\|_{L_q} \|h\|_{L_r}$$

5. Let f be a function in $L_p[0,1]$ for all $p \in [1, P]$. Show that $\|f\|_{L_p}$ is a continuous function of p over the interval $1 \le p \le P$. Consider also the case $P = \infty$.