Classical Analysis – Math 108b

Homework #5

due 11 am Monday, February 14, 2005.

1. If f is Lebesgue measurable, prove that there is a Borel measurable function g such that $f = g$ except, possibly, on a Borel set of measure zero.

2. Let $f : [a, b] \to \bar{\mathbb{R}}$ be measurable and finite a.e. Prove that f is a pointwise limit (a.e.) of polynomials.

3. Let $\{f_n\}$ be a sequence of measurable finite a.e. functions, and $\forall \varepsilon > 0$ there exists an N such that $m\{x \mid |f_n(x) - f_m(x)| \geq \varepsilon\} < \varepsilon$ whenever $m, n > N$. Prove that $\{f_n\}$ converges in measure to some measurable function.

4. Find two decompositions of the real line, $\cup_{n=1}^{\infty} A_n = \mathbb{R} = \cup_{n=1}^{\infty} B_n$, such that

$$\lim_{N\to\infty} \int_{\{\cup_{n=1}^{N} A_n\}} \frac{x}{1+x^2} dx \neq \lim_{N\to\infty} \int_{\{\cup_{n=1}^{N} B_n\}} \frac{x}{1+x^2} dx$$

5. For every pair of measurable functions $f, g : [0, 1] \to \mathbb{R}$ let

$$d(f, g) = \int_{[0,1]} \frac{|f - g|}{1 + |f - g|} dx$$

Prove that a sequence $\{f_n\}$ of measurable functions converges in measure to a measurable function f if and only if $\lim_{n\to\infty} d(f_n, f) = 0$.