

Classical Analysis – Math 108b

Final exam

due 5 pm Tuesday, March 15, 2005.

You can use your notes for this course, your solutions to course homework, and the textbooks. No collaboration or a direct assistance of other people!

All problems are weighted equally.

1. Find a function f such that $f \in L_1([0, 1])$ but $f \notin L_p([0, 1])$ for $p > 1$.
2. Let $f_n : [0, 1] \rightarrow \mathbb{R}$, $n \geq 1$, be a sequence of measurable functions. Show that the following three statements are equivalent:
 - (a) There exists a subsequence $\{f_{n_k}\}_{k \geq 1}$ of $\{f_n\}_{n \geq 1}$ which converges to zero a.e.
 - (b) There exists a sequence $\{c_n\}_{n \geq 1} \subset \mathbb{R}$ such that $\limsup_{n \rightarrow \infty} |c_n| > 0$ and the series $\sum_{n \geq 1} c_n f_n(t)$ converges for almost all $t \in [0, 1]$.
 - (c) There exists a sequence $\{c_n\}_{n \geq 1} \subset \mathbb{R}$ such that $\sum_{n \geq 1} |c_n| = +\infty$ and the series $\sum_{n \geq 1} c_n f_n(t)$ is absolutely convergent for almost all $t \in [0, 1]$.
3. Show that the unit ball in an infinite dimensional Hilbert space contains infinitely many *disjoint* translates of a ball of radius $\frac{1}{4}$.
4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a 2π -periodic C^2 function. Prove that there exists $M > 0$ such that for every $n \in \mathbb{N}$ and every $x \in \mathbb{R}$

$$|s_n(f)(x) - f(x)| \leq \frac{M}{n}.$$

5. Prove the Poincare recurrence theorem: Given a measure preserving map $T : (X, \mu) \rightarrow (X, \mu)$, $\mu(X) < \infty$, then for any measurable set $E \subset X$, $T^n(x) \in E$ infinitely often for almost all $x \in E$.

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Please, spend not more than 3 hours.