Problem 1.

It is easy to see that the tangent vector to the curve $(r(t), \theta(t))$ in polar coordinates has components $\dot{r}$ and $r\dot{\theta}$. So, the length of a curve is

$$l = \int_a^b \sqrt{(\dot{r}(t))^2 + (r(t)\dot{\theta}(t))^2} dt.$$

Using this formula we compute length of the curve $(2\sin\pi t, \pi t)$:

$$l = \int_0^1 \sqrt{4\pi^2\cos^2\pi t + 4\pi^2\sin^2\pi t} dt = 2\pi.$$

Problem 2.

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Then we can construct partition $Q = \{y_1 \ldots, y_n\}$ of $[c, d]$ such that $x_i = \phi(y_i)$. Also, for any partition of $[c, d]$ we can construct corresponding partition of $[a, b]$. Since $g = f \circ \phi$, values of f on $[x_{i-1}, x_i]$ are the same as values of g on $[y_{i-1}, y_i]$. So, we have

$$U(Q, g, \beta) = U(P, f, \alpha), \quad L(Q, g, \beta) = L(P, f, \alpha).$$

The statement follows from these equalities and Theorem 6.6 from [Rudin].

Problem 3.

We define $d(E, F) = m((F \cup E) \setminus (F \cap E))$. Let $d(E, F) = 0$ and $d(F, G) = 0$. First, we see that $F \triangle E = (F \setminus E) \cup (E \setminus F)$. So, $d(E, F) = 0$ is equivalent to $m(E \setminus F) = 0$ and $m(F \setminus E) = 0$. Also, $E \setminus G \subset (E \setminus F) \cup (F \setminus G)$. Since the measure of the last two sets is zero we have $m(E \setminus G) = 0$. By the same argument, $m(G \setminus E) = 0$ and $d(E, G) = 0$. Since the relation $\triangle$ is obviously reflexive and symmetric it is indeed an equivalence relation.

Since $E \setminus G \subset (E \setminus F) \cup (F \setminus G)$ and $G \setminus E \subset (G \setminus F) \cup (F \setminus E)$ then

$$d(E,G) = m((E\setminus G)\cup(G\setminus E)) \leq m((E\setminus F)\cup(F\setminus G)\cup(G\setminus F)\cup(E\setminus G)) \leq d(E,F)+d(F,G).$$

This proves the triangle inequality. Other conditions are trivial.

Problem 4.

First, let $t_1 \leq t_2 \in \mathbb{R}$ then we have $X_{t_1} = \{x \mid f(x) < t_1\} \subset \{x \mid f(x) < t_2\} = X_{t_2}$. So, $\phi(t_1) \leq \phi(t_2)$.

Now, let $t_1 \leq t_2 \leq \ldots \leq t_i \leq \ldots$ be a sequence and $\lim t_i = t$. Then $X_t = \bigcup_i X_{t_i}$ and $m(X) = \lim_{i\to\infty} m(X_{t_i})$. This means that $\lim_{i\to\infty} \phi(t_i) = \phi(t)$ and ϕ is continuous on the left.

Again, $\lim_{t\to\infty} \phi(t) = m(\bigcup_{t\geq t_0}) = 1$.

Problem 5.

Since $f'(x) = \lim_{n\to\infty} n(f(x + 1/n) - f(x))$, it is a limit of a sequence of continuous functions (f is differential and, therefore, continuous). Continuous functions are measurable, also their limit is measurable. So, $f(x)$ is measurable.