Problem 2.

Let $g(x) = f(x+\alpha)$. If $f(x) = \sum_k e^{ikx} f_k$ then $g(x) = \sum_k e^{ikx} e^{ik\alpha} f_k = \sum_k e^{ikx} \tilde{f}_k$, where $\tilde{f}_k = e^{ik\alpha} f_k$. If $f(x) = \sum_k (a_k \sin kx + b_k \cos kx)$ then by the above argument we get $g(x) = \sum_k (\tilde{a}_k \sin kx + \tilde{b}_k \cos kx)$, where $\tilde{a}_k = a_k \cos k\alpha - b_k \sin k\alpha$ and $\tilde{b}_k = a_k \sin k\alpha + b_k \cos k\alpha$.

Problem 3.

Consider function $f(x) = x^2$. Then $\int_{-1}^{1} f(x)dx = 2/9$, $\int_{-1}^{1} f^2(x)dx = 4/5$ and $\int_{-1}^{1} f(x) \cos nx dx = 4(-1)^n/\pi^2 n^2$. Then we apply Parseval equation and get the result.

Problem 4.

We know that $s_n(f)(x) = 1/2\pi \int_{-\pi}^{\pi} f(x+t) \frac{\sin(2n+1)/2t}{\sin t/2} dt$. Then

$$s_n(f)(x_0) - f(x_0) = 1/2\pi \int_{-\pi}^{\pi} (f(x_0+t) - f(x_0)) \frac{\sin(2n+1)/2t}{\sin t/2} dt =$$
$$1/2\pi \int_{-\pi}^{\pi} \frac{f(x_0+t) - f(x_0)}{t} \frac{t}{\sin t/2} \sin(2n+1)/2t dt.$$

Since the derivative exists, function $\frac{f(x_0+t)-f(x_0)}{t}$ is integrable. So, se can apply Riemann's lemma to show that $s_n(f)(x_0) - f(x_0) \to 0$.

Problem 5.

We denote by $k_n(t)$ Fejer's kernel. So, we have from continuity of f at point x:

$$|f(x) - 1/\pi \int_{-\pi}^{\pi} f(x+t)k_n(t)dt| \leq 1/\pi \int_{-\pi}^{\pi} |f(x) - f(x+t)|k_n(t)dt \leq$$
$$\epsilon/\pi \int_{|t|<\delta} k_n(t)dt + 2\|f\|_{C^0}/\pi \int_{\delta \leq |t| \leq \pi} k_n(t)dt.$$

The last integral is small given n is large enough. This implies the result.