Problem 1.

By Cauchy-Shwartz inequality

$$|\int_0^1 f(t)\sin\pi t dt| \leq \int_0^1 |f(t)\sin\pi t| dt \leq$$
$$\left(\int_0^1 |f(t)|^2 dt\right)\left(\int_0^1 |sin\pi t|^2 dt\right) = \frac{1}{\sqrt{2}}\left(\int_0^1 |f(t)|^2 dt\right).$$

Also, equality holds if and only if $f(t) = C\sin\pi t$. Indeed, consider functions f and g. Then we have

$$(f+tg, f+tg) = \|f\|^2 + t^2\|g\|^2 + 2t(f,g) \geq 0.$$

Now, if we put $t = -(f,g)/\|g\|^2$ we get Cauchy-Shwartz inequality. So, for the equality to be true we need $(f+tg, f+tg) = 0$, which means that $f = -tg$ for some t.

Problem 2.

We know that $(f,g) = 1/4(\|f+g\|^2 - \|f-g\|^2)$. So, we have

$$(f,g) = \frac{1}{2}\left(\int_{-1}^1 |x|(f+g)^2 + 3(f'+g')^2 dx - \int_{-1}^1 |x|(f-g)^2 + 3(f'-g')^2 dx\right) =$$
$$\int_{-1}^1 |x|(fg) + 3(f'g')dx.$$

Now, apply Cauchy-Shwartz inequality to functions f and x^2:

$$|(f,x^2)| = |\int_{-1}^1 |x|^3 f + 6xf' dx| \leq \|f\|\|x^2\| =$$
$$\left(\int_{-1}^1 |x|f^2 + 3|f'|^2 dx\right)^{1/2}\left(\int_{-1}^1 |x|^3 + 12x^2 dx\right)^{1/2} = \left(\int_{-1}^1 |x|f^2 + 3|f'|^2 dx\right)^{1/2}\frac{5}{\sqrt{3}}.$$

Problem 3.

Since continuous functions are dense in $L_2[0,1]$ and polynomials are dense in $C[0,1]$ we conclude that polynomials are dense in $L_2[0,1]$. Since $L_2[0,1]$ is separable, we can choose a countable dense set of polynomials. From this set we can choose a linear independent set of polynomials and apply the Gramm-Schmidt process. In the process we use only addition and multiplication by a constant, so the result will by also a set of polynomials.

Problem 4. Let $x \in C[0,1]$, $c \in [0,1]$ and $T(x) = x(c)$. Then if $|x(t)| \leq A$ then also $T(x) \leq A$. So, functional T is bounded and, therefore, continuous with respect to the norm of $C[0,1]$. Now, let

$$x_n(t) = \begin{cases} n, & \text{if } t \in [c - 1/2n^2, c + 1/2n^2] \\ 0, & \text{if } t \notin [c - 1/2n^2, c + 1/2n^2]. \end{cases}$$

Then $\int |x_n|^2 = 1$ and $x_n(c) = n$. We can approximate x_n by a continuous function $\tilde{x}_n$ in L_2-norm in such a way that $\tilde{x}_n(c) = n$. Then $T(\tilde{x}_n) = n$. So, functional T is NOT bounded with respect to L_2-norm.

Problem 5.

Define map $f : l^2_{\mathbb{Z}} \to l^2$ by formula

$$f : (\ldots, x_{-1}, x_0, x_1, \ldots) \to (x_0, x_{-1}, x_1, \ldots).$$

This map is one-to-one. Also it is linear. Obviously it preserves the norm. So, it is an isomorphism