Problem 1.

By Fatou theorem,

$$\int_{\mathbb{R}} f + g \leq \liminf_{n\to\infty} \int_{\mathbb{R}} f_n + g_n,$$

which implies

$$\int_{\mathbb{R}} f \leq \liminf_{n\to\infty} \int_{\mathbb{R}} f_n.$$

Also, since $g_n - f_n \geq 0$,

$$\int_{\mathbb{R}} g - f \leq \liminf_{n\to\infty} \int_{\mathbb{R}} g_n - f_n$$

and

$$-\int_{\mathbb{R}} f \leq \liminf_{n\to\infty} (-\int_{\mathbb{R}} f_n).$$

These two inequalities imply the result.

Problem 2.

Notice that $\int_{\mathbb{R}} f(2^n + 1/n) = 1/2^n \int_{\mathbb{R}} f(x)$. Then $\int_R \sum_{n=1}^N f(2^n + 1/n) = \sum_{n=1}^N 1/2^n \int_{\mathbb{R}} f(x)$. Also, $\int_R \sum_{n=1}^N |f(2^n + 1/n)| = \sum_{n=1}^N 1/2^n \int_{\mathbb{R}} |f(x)|$. Since $\int_R \sum_{n=1}^\infty |f(2^n+1/n)| = \sum_{n=1}^\infty \int_{\mathbb{R}} |f(2^n+1/n)|$ and $|\sum_{n=1}^N f(2^n+1/n)| \leq \sum_{n=1}^N |f(2^n+1/n)|$ we have $\int_R f = \int_{\mathbb{R}} \phi$

Problem 3.

We have

$$|f * g(x)| = |\int_{\mathbb{R}} f(x-y)g(y)dy| \leq \int_{\mathbb{R}} |f(x-y)g(y)|dy \leq$$
$$\|g\|_\infty \int_R |f(x-y)|dy = \|g\|_\infty \|f\|_1.$$

So, the convolution is bounded.

Now,

$$|f * g(x_1) - f * g(x_2)| = |\int_{\mathbb{R}} (f(x_1 - y) - f(x_2))g(y)dy| \leq$$
$$\int_{\mathbb{R}} |(f(x_1 - y) - f(x_2 - y))g(y)|dy \leq \|g\|_\infty \int_R |(f(x_1 - y) - f(x_2 - y))|dy.$$

The last integral tends to 0 when $x_2 \to x_1$. Indeed, we can approximate function f in L_1 norm by continuous functions, i.e. there exists a function g such that $\|f - g\|_1 < \epsilon$. Consider an interval $[-T, T]$ such that $\int_{-T}^T |f|dx < \|f\| - \epsilon$ and $\int_{-T}^T |g|dx < \|f\| - \epsilon$. On this interval g is uniformly continuous and there is such δ_0 that $|g(x-\delta) - g(x)| < \epsilon/2T$ for all $0 \leq \delta < \delta_0$. This means that $\int_{-T}^T |g(x-\delta) - g(x)| < \epsilon$. And from triangle inequality we get $\int_{-T}^T |f(x-\delta) - f(x)| < 2\epsilon$. Since the integral of f by $\mathbb{R} \subset [-T, T]$ is less than ϵ we get $\|f(x-\delta) - f(x)\|_1 \to 0$ when $\delta \to 0$.

So, the convolution is continuous.

Problem 4.

Since $f \in L_p$ and $g \in L_q$ we now that $f^s \in L_{p/s}$ and $g^s \in L_{q/s}$. Now, let $s = \frac{pq}{p+q}$. Then $s/p + s/q = 1$ and we can apply Holder inequality:

$$\int_{\mathbb{R}} |f^s g^s| \le \|f\|_p^s \|g\|_q^s.$$

So, $fg \in L_s$. Since $1/s + 1/r = 1$ we can apply Holder inequality to functions fg and h:

$$\int_{\mathbb{R}} |fgh| \le \|fg\|_s \|h\|_r$$

and using previous inequality we get

$$\|fg\|_s \|h\|_r \le \|f\|_p \|g\|_q \|h\|_r.$$

Problem 5.

Consider function $F(p) = \log \int_0^1 |f(x)|^p dx$. Let's show that this function is convex, i.e. for any $p_1 < p_2$ and $\lambda \in [0,1]$ we have inequality

$$F(p_1(1-\lambda) + p_2\lambda) \le F(p_1)(1-\lambda) + F(p_2)\lambda.$$

Indeed, $F(p_1(1-\lambda) + p_2\lambda) = \log \int_0^1 |f|^{p_1(1-\lambda)} |f|^{p_2\lambda}$. Since $|f|^{p_1(1-\lambda)} \in L_{1/(1-\lambda)}$ and $|f|^{p_2\lambda} \in L_{1\lambda}$ we can apply Holder inequality:

$$F(p_1(1-\lambda) + p_2\lambda) = \log \int_0^1 |f|^{p_1(1-\lambda)} |f|^{p_2\lambda} \le$$
$$\log \left(\left(\int_0^1 |f|^{p_1} \right)^{1/(1-\lambda)} \left(\int_0^1 |f|^{p_2} \right)^{1/\lambda} \right) = 1/(1-\lambda) \log \|f\|_{p_1}^{p_1} + 1/\lambda \log \|f\|_{p_2}^{p_2}.$$

This proves convexity. Convexity implies continuity, since the right hand side of the inequality $F(p_1(1-\lambda) + p_2\lambda) \le F(p_1)(1-\lambda) + F(p_2)\lambda$ tends to $F(p_1)$ when $\lambda \to 0$ and function F is nonnegative.

Now, let's prove that $\|f\|_p \to \|f\|_\infty$ when $p \to \infty$. First, suppose that $\|f\|_\infty = 1$. Then for any $\epsilon > 0$ there exists a set of X measure δ such that $|f(x)| > 1 - \epsilon$ when $x \in X$. So, $|f|^p > (1-\epsilon)^p$ on X and $\int_0^1 |f|^p dx > \delta(1-e)^p$. Thus, $\|f\|_p > \delta^{1/p}(1-\epsilon)$. The last quantity tends to $1-\epsilon$ when $p \to \infty$. And since $\|f\|_p \le 1$ we have $\lim_{p\to\infty} \|f\|_p = 1$. If $\|f\|_\infty \ne 1$ consider function $\tilde{f} = f/\|f\|_\infty$. Then $\|\tilde{f}\|_p \to 1$, which means that $\|f\|_p \to \|f\|_\infty$.