Problem 1.

Since f is measurable, $f^{-1}([a,\infty)) = F_a$ is measurable for any a. Then $F_a = B_a \cup N_a$, where B_a is a Borel set and N_a has measure 0. But $N_a \subset N_a^*$, where N_a^* is a Borel set of measure 0. Now, let's $N = \cup_{a\in\mathbb{Q}} N_a^*$, then $m(N) = 0$. Let $g = f\chi_{N^c}$. Then g is Borel measurable. Indeed, for $0 < a \in \mathbb{Q}$ we have $g^{-1}([a,\infty)) = B_a \setminus N$ and for $0 > a \in \mathbb{Q}$ $g^{-1}([a,\infty)) = B_a \cup N$. Since B_a and N are Borel sets, $g^{-1}([a,\infty))$ is a Borel set for any rational a. Taking limit we see that it is a Borel for any real a and function g is Borel measurable.

Problem 2.

By Lusin Theorem, for any n there exists a continuous function g_n such that $m(\{x \mid f \neq g_n\}) < 1/n$. Since polynomials are dense in $C[a,b]$, there exists a polynomial p_n such that $|p_n(x) - g_n(x)| \leq 1/n$ for any $x \in [a,b]$. By construction of the sequence g_n, it converges to f in measure, so there exists a subsequence q_{n_k} converging to f a.e. (see Problem 3). And since $\|p_{n_k} - g_{n_k}\| \to 0$ when $k \to \infty$ we see that p_{n_k} converges to f a.e..

Problem 3.

First, we show that there is a subsequence f_{n_k} converging to some function f a.e.. To do it we choose f_{n_k} to satisfy

$$m(E_k) < 2^{-k}, \quad \text{where } E_k = \{x \mid |f_{n_{k+1}} - f_{n_k} \geq 2^{-k}|\}.$$

Then, since $\sum_k m(E_k) < \infty$, by Borel-Cantelli lemma we have $m(E) = 0$, where $E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$. For any $x \notin E$ we have $x \notin \bigcup_{j=k}^{\infty} E_j$ for some k. So, $|f_{n_{j+1}}(x) - f_{n_j}(x)| < 2^{-j}$. This means, that the sequence $f_{n_j}(x)$ converges. So, the sequence f_{n_j} converges a.e. to some function f.

Since $f - f_{n_k} = \sum_{j=k}^{\infty}(f_{n_{j+1}} - f_{n_j})$ we have

$$|f(x) - f_{n_k}(x)| \leq \sum_{j=k}^{\infty} |f(x)_{n_{j+1}(x)} - f_{n_j}(x)| < 2^{-k+1}$$

for any $x \notin \bigcup_{j=k}^{\infty} E_j$. So,

$$m(\{x \mid |f(x) - f_{n_k}(x)| \geq 2^{-k+1}\}) \leq m(\bigcup_{j=k}^{\infty} E_j) \leq 2^{k+1}.$$

We see that f_{n_k} converges to f in measure, so f_n converges to f too, since it is Cauchy in measure.

Problem 4.

Since $\int_a^b \frac{x}{1+x^2}dx = \frac{1}{2}\log\frac{1+a^2}{1+b^2}$, we have

$$\lim_{n\to\infty} \int_{[-n,n]} \frac{x}{1+x^2}dx = 0$$

and

$$\lim_{n\to\infty} \int_{[-n,2n]} \frac{x}{1+x^2}dx = \log 2.$$

Problem 5.

Assume that f_n converges to f in measure. Then for any ϵ and δ there exits N such that $m(\{|f_n - f| > \delta\}) < \epsilon$ for $n > N$. Then

$$d(f_n, f) = \int_{\{|f_n - f| > \delta\}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{\{|f_n - f| \leq \delta\}} \frac{|f_n - f|}{1 + |f_n - f|} \leq \epsilon M + \delta,$$

where $M = \sup \frac{|f_n - f|}{1+|f_n - f|}$. Choosing ϵ and δ small enough we have $d(f_n, f) \to 0$.

Assume $d(f_n, f) \to 0$. Since $\int \frac{|f_n - f|}{1+|f_n - f|} dx \to 0$ and $\frac{|f_n - f|}{1+|f_n - f|} \geq 0$ we have $\frac{|f_n - f|}{1+|f_n - f|} \to 0$ a.e.. So, $f_n \to f$ a.e. and $f_n \to f$ in measure.