Problem 1.

The statement is NOT true. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} .$$

Then f is not integrable, but $f = 0$ a.e.. Moreover, $f = 0$ except at countably many points. But if $f = g$ except at finitely many points and g is integrable then f is also integrable. Indeed, changing f at finitely many points we can add at most a finite number of discontinuities which has measure zero.

Problem 2.

Let $f : D \to \mathbb{R}$ and D is measurable. Suppose a set $X_\alpha = \{x \mid f(x) > \alpha\}$ is measurable for any rational α. Let's show that X_α is measurable for any real α. Let's pick any $\alpha \in \mathbb{R}$ and a sequence $\alpha_i \in \mathbb{Q}$, $\alpha_{i-1} > \alpha_i > \alpha$ such that $\alpha_i \to \alpha$ when $i \to \infty$. Then we have $X_{\alpha_1} \subset X_{\alpha_2} \subset \ldots \subset X_{\alpha_n} \subset \ldots$ and $X_\alpha = \cup X_{\alpha_i}$. The last set is measurable by Theorem 11.3. So, X_α is measurable for any real α and f is measurable.

Problem 3.

By Theorem 11.17 functions $g(x) = \liminf_{n\to\infty} f_n(x)$ and $h(x) = \limsup_{n\to\infty} f_n(x)$ are measurable. The set of points at which f_n converges is exactly the set where $g - h = 0$. Since $g - h$ is measurable, this set is measurable.

Problem 4.

Let $f = 0$ a.e. and f be integrable. So, $f \neq 0$ on some set E of measure zero. We can cover the set E by intervals (a_i, b_i) of total length $\leq \varepsilon$. Then $\int_a^b f(x)dx = \sum_i \int_{a_i}^{b_i} f(x)dx \leq \sup f(x)\varepsilon$. Since ε is arbitrary the integral is zero.

It $f = g$ a.e. then $f - g = 0$ a.e. and $\int_a^b (f - g)dx = 0$.

Problem 5.

We can assume that $0 < m(E) < \infty$. Suppose there exists such ε that for any interval I we have $\varepsilon m(I) > m(I \cap E)$. Then for any covering of the set E by intervals $\{I_i\}$ we have

$$\sum m(I_i) \geq 1/\varepsilon \sum m(I_i \cap E) \geq \frac{1}{\varepsilon} m(E).$$

But there is a covering of E by intervals of total length less than $m(E) + \delta$ for any δ. Taking $\delta < (1/\varepsilon - 1)m(E)$ we get a contradiction.