Problem 1.

Assume that $\int_1^\infty f(x)dx$ converges. Since $f(x) \geq 0$ and $f(x)$ decreases monotonically on $[1, \infty)$ we have inequality

$$0 \leq S_N = \sum_{n=2}^{N} f(n) \leq \int_1^N f(x)dx.$$

The integral converges, so the sequence S_N is bounded from above. Since this sequence is nonnegative it has a limit when $n \to \infty$. So, the series converge.

Now assume that $\sum_{n=1}^\infty f(n)$ converges. Again it is easy to see that

$$0 \leq F(A) = \int_1^A f(x)dx \leq \sum_{n=1}^{[A]} f(n).$$

Since the series converge, $F(A)$ is bounded from above. But $F(A)$ is increasing since $f(x) \geq 0$. So, $F(A)$ has a limit when $A \to \infty$.

Problem 2.

We know that if $\alpha(x) = I(x-c)$ then $\int_a^b f(x)d\alpha = f(c)$, where

$$I(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

From this and linearity of the integral we have:

a)

$$\alpha = \begin{cases} 0, & \text{if } x = a, \\ 1/2 & \text{if } a < x < b \\ 1 & \text{if } x = b \end{cases},$$

b) If $a, b \notin \{\frac{1}{n}, n \in \mathbb{N}\}$ then $\alpha = \sum_1^\infty \frac{1}{2^n} I(x - 1/n)$,

c)

$$\alpha = \begin{cases} x, & \text{if } x \in [a, (b+2a)/3] \\ (b+2a)/3 & \text{if } x \in [(b+2a)/3, (2b+a)/3] \\ x - (b-a)/3 & \text{if } x \in [(2b+a)/3, b] \end{cases}$$

d) $\alpha = \sum_{n=1}^\infty \alpha_n$, where

$$\alpha_n = \begin{cases} 0, & \text{if } a \leq x < a + \frac{b-a}{2^n}, \\ \frac{1}{n}x - \frac{1}{n}\left(a + \frac{b-a}{2^n}\right), & \text{if } a + \frac{b-a}{2^n} \leq x < a + \frac{b-a}{2^{n-1}} \\ \frac{1}{n}\frac{b-a}{2^{n-1}} & \text{if } a + \frac{b-a}{2^{n-1}} \leq x \leq b. \end{cases}$$

Problem 3.

To find the length of a curve $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ we use the formula $l(\gamma) = \int_0^1 \sqrt{(\gamma_1')^2 + \cdots + (\gamma_n')^2} dt$. So, we have:

a) $l = \int_0^1 \sqrt{1 + 4\pi^2} dt = \sqrt{1 + 4\pi^2}$,

b) $l = \int_0^1 2\pi dt = 2\pi$,

c) $l = \int_0^1 4\pi dt = 4\pi$.

Problem 4.

Suppose the set E is covered by intervals (x_i, y_i), $i = 1, \ldots, n$ and $\sum |x_i - y_i| \leq m^*(E) + \delta$. Then the set $f((x_i, y_i))$ lies in an interval (x_i', y_i') such that $|x_i' - y_i'| \leq K|x_i - y_i|$. Indeed, for any two points $x, y \in (x_i, y_i)$ we have

$$|f(x) - f(y)| \leq K|x - y| < K|x_i - y_i|.$$

So, the distance between any two points from the set $f((x_i, y_i))$ is not greater than $K|x_i - y_i|$.

So, the set $f(E)$ is covered by intervals (x'_i, y'_i) and we have inequality

$$\sum |x'_i - y'_i| \leq K \sum |x_i - y_i| \leq K(m^*(E) + \delta).$$

Since δ is arbitrary we have

$$m^*(f(E)) \leq \sum |x'_i - y'_i| \leq K \sum |x_i - y_i| \leq K m^*(E).$$

Problem 5.

The set E is contained in the set F of all reals in $[0,1]$ whose decimal expansion doesn't contain 2's. Let $x = \sum_{n=1}^{\infty} 10^{-n} x_n$, so $0.x_1 x_2 \ldots$ is the decimal expansion of x. Then,

$$F = [0,1] \setminus \bigcup_{n=1}^{\infty} F_n,$$

where

$$F_n = \{x = \sum_{n=1}^{\infty} 10^{-n} x_n \mid x_1, \ldots, x_{n-1} \neq 2,\ x_n = 2\}.$$

It's easy to see that $m(F_n) = (9/10)^{n-1} 1/10$. Since $F_n \cap F_m = \emptyset$ if $n \neq m$ then

$$m(F) = 1 - \frac{1}{10} \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^{n-1} = 0.$$