

Problem 1.

The statement is NOT true. Consider function $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ -1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.

Clearly $f^2(x) = 1$ is integrable, but f is not integrable. If f^3 is integrable then we can apply Theorem 6.11 from Rudin with $\phi(t) = t^{1/3}$ and get that f is indeed integrable.

Problem 2.

From the definition of the Cantor set in 2.44 we know that it can be covered by a finite set of intervals of arbitrarily small total length. Denote these intervals by $[u_j, v_j]$. We can choose them in such a way that $E \subset \cup(u_j, v_j)$. Now, the set $K = [a, b] \setminus \cup(u_j, v_j)$ is compact and f is uniformly continuous on it, i.e. there exists $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ for all $s, t \in K$, $|s - t| < \delta$. Now we form a partition P such that all u_j and v_j occur in it and no point from any (u_j, v_j) occurs in P . Then we have

$$U(P, f) - L(P, f) \leq C\epsilon,$$

and Theorem 6.6 implies that f is integrable.

Problem 3.

Since

$$\|f + g\|^2 \leq \int_a^b |u + v|d\alpha = \int_a^b |f|^2 + |g|^2 + 2|f||g|d\alpha,$$

by the Schwartz inequality we have

$$\|f + g\|^2 \leq \int_a^b |f|^2 + |g|^2d\alpha + 2\sqrt{\int_a^b |f|^2d\alpha \int_a^b |g|^2d\alpha} = (\|f\| + \|g\|)^2.$$

So, we have $\|f + g\| \leq \|f\| + \|g\|$.

Problem 4.

Let $\|f\| = \int_a^b |f|dx$. Then $\|f\| \geq 0$. Let $\|f\| = 0$. Then, by Theorem 6.20 we have $F(x) = \int_a^b x|f(t)|dt$ is differentiable on $[a, b]$ and $F'(x) = f(x)$. But since $|f|$ is positive we have $0 \leq F(x) \leq F(b) = 0$, i.e. $F(x) = 0$. So, $f(x) = 0$. Now, we need to check the triangle inequality:

$$\|f + g\| = \int_a^b |f + g|dx \leq \int_a^b |f| + |g|dx = \|f\| + \|g\|.$$

So, $\|\cdot\|$ is indeed a norm on $C[a, b]$.

But it does NOT define a norm on $R[a, b]$. Indeed a function $f(0) = 1$, $f(x) = 0$ for $x \in (0, 1]$ is integrable on $[0, 1]$, but $\|f\| = 0$.

Problem 5.

Let

$$\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } x \geq 0, \end{cases}$$

and

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0, \end{cases}.$$

Then $\int_{-1}^1 |f|d\alpha = 0$ and α is continuous.