Problem 1.

Since $\|f(t)\| = 1$ we have $f(t) \cdot f(t) = 1$. Taking derivatives of both sides of the last equality we have: $2f'(t) \cdot f(t) = 0$, which implies the result. Geometrically this means that at any point of the curve $f(t)$ it's speed is orthogonal to the vector $f(t)$.

Problem 2.

(a) Since $x^8 - 2x^4y^2 + y^4 = (x^4 - y^2)^2 \geq 0$ we have $(x^4 + y^2)^2 = x^8 + y^4 + 2x^4y^2 \geq 4x^4y^2$.

This inequality implies continuity of the function $h(x, y) = \frac{4x^6y^2}{(x^4+y^2)^2}$. It follows from the inequality

$$0 \leq \frac{4x^6y^2}{(x^4 + y^2)^2} \leq x^2$$

that $\lim_{x\to 0,\, y\to 0} h(x, y) = 0$.

This means that the function $f(x, y)$ is continuous at the origin. Continuity at other points is trivial.

(b) We define $g_\theta(t) = f(t\cos\theta, t\sin\theta$. So, $g_\theta(0) = f(0, 0) = 0$. Also we have

$$g_\theta(t) = t^2 - t^3\cos\theta\sin 2\theta - \frac{4t^4}{t^4\cos^8\theta + 2t^2\cos^4\theta\sin^2\theta + \sin^4\theta}.$$

Now, derivative of $t^2 - \cos\theta\sin 2\theta$ at the origin is 0. If $\sin\theta \neq 0$ then

$$\frac{1}{t}\frac{4t^4}{t^4\cos^8\theta + 2t^2\cos^4\theta\sin^2\theta + \sin^4\theta} \to 0, \quad \text{when } t \to 0,$$

which means that derivative is 0 (by the definition of derivative). If $\sin\theta = 0$ then

$$\frac{4t^4}{t^4\cos^8\theta + 2t^2\cos^4\theta\sin^2\theta + \sin^4\theta} = \frac{1}{\cos^8\theta} = const,$$

and it's derivative is also 0. So, we have $g'_\theta(0) = 0$.

Now we compute the second derivative:

$$g''_\theta(t) = 2 - 6t\cos\theta\sin 2\theta - \left(\frac{16t^3(t^2\cos^4\theta + \sin^2\theta)^2 - 16t^5(t^2\cos^4\theta + \sin^2\theta)}{(t^2\cos^4\theta + \sin^2\theta)^4}\right)'.$$

And it's easy to see by the method above that the derivative of the last expression is 0 at the origin. So, $g''_\theta(0) = 2$.

(c) It's easy to compute:

$$f(x, x^2) = x^2 + x^4 - 2x^2x^2 - \frac{4x^6x^4}{(x^4 + x^4)^2} = x^2 - x^4 - x^2 = -x^2.$$

This means that $(0, 0)$ is not a local minimum, since $f(x, x^2) \leq f(0, 0) = 0$ for any x and we can choose x to be close to 0.

Problem 3.

(a) The range of f is $\mathbb{R} \setminus \{(0, 0)\}$. Since $f_1^2 + f_2^2 = e^{2x} > 0$ it doesn't contain the origin. In the part (c) we will see that it contains all other points.

(b) The Jacobian is

$$\det\begin{pmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{pmatrix} = e^{2x} > 0.$$

(c) Let $a = (0, \pi/3)$, $b = f(a)$. Then $g = (g_1, g_2)$, where

$$g_1(f_1, f_2) = \frac{1}{2}\log(f_1^2 + f_2^2),$$

$$g_2(f_1, f_2) = \arctan(f_2/f_1).$$

Now, we compute

$$f'(0, \pi/3) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

and

$$g'(f_1, f_2) = \begin{pmatrix} \frac{2f_1}{2(f_1^2+f_2^2)} & \frac{2f_2}{2(f_1^2+f_2^2)} \\ \frac{-f_1/f_2^2}{1+f_2^2/f_1^2} & \frac{1/f_1}{1+f_2^2/f_1^2} \end{pmatrix}.$$

So,

$$g'(f(0, \pi/3)) = g'(1/2, \sqrt{3}/2) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix},$$

and equation (52) in Rudin can be verified by multiplication of these matrices.

(d) The image of a line $x = a$ is a set $\{(e^a \cos y, e^a \sin y) \mid y \in \mathbb{R}\}$, which is a circle centered at 0 of radius e^a.

The image of a line $y = b$ is a set $\{(e^x \cos b, e^x \sin b) \mid x \in \mathbb{R}$, which is a ray through the origin with slope $\tan b$.

Problem 4.

We define:

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Then $f(0, 1, -1) = 1 - 1 = 0$. And $D_1 f = 2xy_1 + e^x$, so $(D_1 f)(0, 1, -1) = 1 \neq 0$. The Implicit Function Theorem implies now existence of a function g such that $f(g(y_1, y_2), y_1, y_2) = 0$. Also $g(1, -1) = 0$. So, by the Implicit Function Theorem we have

$$(D_1 g)(1, -1) = -D_2 f / D_1 f = 0,$$
$$(D_2 g)(1, -1) = -D_3 f / D_1 f = -1.$$

Problem 5.

The determinant is a polynomial mapping $\det : \mathbb{R}^{n^2} \to \mathbb{R}$, so it's clearly C^1-smooth. Also, a partial derivative of this map by any coordinate a_{ij} is a corresponding cofactor. So, if we consider matrices with $\det = 1$ there exists a nonzero partial derivative. This means, that the rank of the map is 1 and we can apply the corollary to the Implicit Function Theorem.