Problem 1.

Consider function
$$f(x) = \frac{1}{x(1-|\log x|)^2}.$$
Then $\int_t^1 f(x)dx = 1 - 1/(1+\log t) \to 1$ when $t \to 0$. So, $f(x) \in L_1$. But for any $p > 1$ we can find $M > 0$ such that $\frac{1}{x^p(1-\log x)^{2p}} \geq M/x$. So, $f \notin L_p$ for any $p > 1$.

Problem 2.

$(a) \Rightarrow (b),(c)$: We can assume that $f_n(t) \to 0$ a.e.. Then by Egoroff theorem f_n converges uniformly outside of a arbitrarily small set. So, we can find a subsequence f_{n_k} such that $|f_{n_k}| < 1/2^k$ on a set C_k, where $m(C_k) > 1 - 1/k$. Now set $c_n = 1$ for $n \in \{n_k\}$ and $c_n = 0$ for $n \notin \{n_k\}$. Then, since $m(\cup C_k) = 1$, series $\sum_k c_k f_k(t)$ converges absolutely a.e..

$(b) \Rightarrow (a)$: Since $\limsup |c_n| > 0$ we can find a subsequence c_{n_k} such that $|c_{n_k}| > a$. Since the series converges we have $f_{n_k} c_{n_k} \to 0$ a.e., which implies the result.

$(c) \Rightarrow (a)$: Suppose (a) is not true. Then there is a set of positive measure, where $|f_n| \geq a > 0$. So, we have $\sum |c_n f_n| > a \sum |c_n|$. The last series diverges, so we have a contradiction.

Problem 3.

Consider an orthonormal basis e_i and balls of radius $1/4$ centered at $3/4e_i$. Clearly, this balls are in the unit ball. Also, the distance between the centers is greater than $1/2$, so these balls are disjoint.

Problem 4.

The Fourier coefficients of f'' are $-k^2 a_k$ and $-k^2 b_k$, where a_k and b_k are Fourier coefficients of f. So, we have $\sum k^4 a_k^2 < \infty$ and $\sum k^4 b_k^2 < \infty$. Then
$$|s_n - f| = |\sum_{k=n}^{\infty} a_k \cos kx + b_k \sin kx|$$
and
$$n|s_n - f| = n|\sum_{k=n}^{\infty} a_k \cos kx + b_k \sin kx| \leq \sum(|ka_k| + |kb_k|) \leq M.$$
This implies the result.

Problem 5.

Consider sets $E \cap \bigcap_{n=m}^{\infty} T^{-n}(X \setminus E) = A_m$. Then sets $T^{-(m+k)}(A_m)$ are disjoint for different k. Indeed, if $x \in T^{-(m+k)}(A_m)$ then $x \in T^{-(m+k)(E)}$, so $x \notin \bigcap_{n=m}^{\infty} T^{-(n+j)}(X \setminus E)$ for $j < k$. So, $x \notin T^{-(m+j)}(A_m)$. Since sets $T^{-(m+k)}(A_m)$ are disjoint and T is measure-preserving, $m(A_m) = 0$. Then $m(\bigcup A_m) = 0$. This implies the result.