

Classical Analysis – Math 108b

Final exam

due 5 pm Wednesday, March 15, 2006 (no other time limits).

1. (10) Real numbers in the interval $[0, 1]$ can be represented by a binary extension $x = 0.d_1d_2d_3d_4\dots$. Unless x is a rational number with denominator of the form 2^k , this representation is unique. Let $f : [0, 1] \rightarrow [0, 1]$ be the function which places an extra zero between each digit:

$$f(0.d_1d_2d_3d_4\dots) = 0.d_10d_20d_30d_40\dots$$

a) Show that f is a measurable function. Note that the value of f is sometimes undefined, like if $x = \frac{1}{2}$. Why isn't this a problem?

b) Find $\int_0^1 f(x)dx$.

2. (10) Suppose a bounded function $f : [0, 1] \rightarrow \mathbb{R}$ has a closed nowhere dense set of discontinuities.

a) Is it true that f must be Riemann integrable?

b) Prove that there exists a homeomorphism $h : [0, 1] \rightarrow [0, 1]$ such that $f \circ h$ is Riemann integrable;

c) Is it true that if for some bounded function $f : [0, 1] \rightarrow \mathbb{R}$ there exists a homeomorphism $h : [0, 1] \rightarrow [0, 1]$ such that $f \circ h$ is Riemann integrable, then the set of discontinuities of f is a closed nowhere dense set?

d)*(+2 extra points) Can you describe (in terms of the set of discontinuities) the set of bounded functions $f : [0, 1] \rightarrow \mathbb{R}$ such that the composition $f \circ h$ is Riemann integrable for some homeomorphism $h : [0, 1] \rightarrow [0, 1]$?

3. (10) Let $\{e_n\}_{n=1}^\infty$ be a countable orthonormal basis in a separable Hilbert space H . Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Consider a linear map $T : H \rightarrow H$ defined by

$$Te_n = e_{\pi(n)}.$$

For every vector $v \in H$, $v = \sum_{n=1}^\infty v_n e_n$ define $Tv = \sum_{n=1}^\infty v_n Te_n = \sum_{n=1}^\infty v_n e_{\pi(n)}$.

a) Show that T is a continuous linear operator;

b) Prove that for every $x \in H$ there exists a limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} T^k x \in H$$

c) Take $x \in H$, denote $z = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} T^k x$. Prove that $Tz = z$.

4. (10) a) Find a function f such that $f \in L_1([0, 1])$ but $f \notin L_p([0, 1])$ for $p > 1$;
- b) For every $1 \leq p < q \leq \infty$ find a function $f \in L_p[0, 1]$ such that $f \notin L_q[0, 1]$;
- c) Can you give these examples for $L_p(\mathbb{R})$?
5. (10) Let $f \in C^{2\pi}$ and assume that $f'(x_0)$ exist. Prove that $s_n(f)(x_0) \rightarrow f(x_0)$ as $n \rightarrow \infty$.