

Problem 1. Since binary expansion is not unique at binary rational points, the function is not well-defined there. We will think that this function is not defined at binary rational points. These points form a countable set, so its measure equals zero.

Now, let's denote $f_i(0.d_1d_2\dots) = d_i2^{-(1+2i)}$. Then we have

$$f(x) = f_0(x) + f_1(x) + \dots$$

Each of functions f_i is measurable, so f is measurable as a limit of measurable functions. Also, $\int_0^1 f(x)dx = \sum \int_0^1 f_i(x)dx$, by Monotone Convergence Theorem. But $\int_0^1 f_i(x)dx = \frac{1}{2}2^{-(1+2i)}$. Then, computing the sum we get the result $\int_0^1 f(x)dx = \frac{1}{3}$.

Problem 2. a) The answer is NO. Indeed, we can easily construct a closed nowhere dense set: if $\{q_i\}$ is the set of rationals in $[0, 1]$, then $I = \cup(q_i - \varepsilon2^{-i}, q_i + \varepsilon2^{-i})$ is an open dense set of measure less than ε . So, $J = [0, 1] \setminus I$ is a closed nowhere dense set of positive measure. Since I is dense, the function χ_J is discontinuous at any point of J . And since I is open χ_J is continuous at any point of I . So, it has closed nowhere set of discontinuities which has positive measure, so it is not Riemann integrable.

b) Let f has a closed nowhere dense set D of discontinuities, where $\mu(D) > 0$. Since D is nowhere dense, $\mu(D) < 1$. Let's denote

$$h(x) = \frac{1}{1 - \mu(D)} \int_0^x \chi_{D^c}(s)ds,$$

where $D^c = [0, 1] \setminus D$. This function is clearly continuous by properties of integral. Also, it's monotone. Indeed, since D is closed nowhere dense, $D^c \cap [x_1, x_2]$ contains an interval, so

$$\int_{x_1}^{x_2} \chi_{D^c}(s)ds > 0.$$

And since $h(0) = 0$ and $h(1) = 1$, $h(x)$ is a homeomorphism of the interval.

Let's show that $\mu(h(D)) = 0$. Since D is closed, $D^c = \cup I_i$, where I_i 's are pairwise disjoint intervals. Also, $h(I_i)$ is an interval of length

$$\mu(h(I_i)) = \frac{1}{1 - \mu(D)} \int_{I_i} \chi_{D^c}(s)ds = \frac{1}{1 - \mu(D)} \mu(I_i),$$

since $I_i \subset D^c$. Taking sum by all intervals we get $\mu(h(D^c)) = 1$, so $\mu(h(D)) = 0$.

c) No, counterexample is $f(x) = \frac{\chi_{\{1/n\}}(x)}{n}$. The set of discontinuities is $\{1/n\}$. It has measure zero, but it's not closed.

d) There exists such homeomorphism h that $f \circ h$ is Riemann integrable if and only if the set of discontinuities D of the function f is of first category, i.e. it is a countable union of closed nowhere dense sets.

To prove this, we will use the following construction. For some intervals $I = (a, b)$, $J = (c, d)$ and a nowhere dense set $D \subset I$, the function

$$h_{(a,b),(c,d),D}(x) = c + \frac{d - c}{(b - a) - \mu(D_i)} \int_a^x \chi_{D^c}(s)ds$$

define a homeomorphism between (a, b) and (c, d) such that $\mu(h(D)) = 0$. This is a slight generalization of the part b).

Now, let's $D = \cup D_i$, where $D_1 \subset D_2 \subset \dots$ are closed nowhere dense. First, we define a homeomorphism g of D to some set of measure zero. We do it inductively. Let $g_1 = h_{(0,1),(0,1),D_1}$, defined on D_1 only. Now, suppose we have g_i defined on

D_i . We define g_{i+1} on D_{i+1} in the following way. Since D_i is closed, D_i^c is open, so $D_i^c = \cup I_j$, where $I_j = (a_j, b_j)$ are pairwise disjoint intervals. We can choose I_j 's to be maximal, i.e. $\partial I_j \subset D_i \cup \{0, 1\}$. Then we define g_{i+1} on each I_j :

$$g_{i+1}(x) = h_{(a_j, b_j), (c_j, d_j), D_i \cap I_j}, \quad \text{when } x \in D_{i+1} \setminus D_i,$$

where

$$c_j = g_i(a_j), \quad d_j = g_i(b_j),$$

and

$$g_{i+1}(x) = g_i(x), \quad \text{when } x \in D_i.$$

Then, from the definition, this map is monotone and continuous on D_{i+1} . Repeating this procedure, we get a continuous monotone map g on \bar{D} . And $g(D) = \cup d(D_i)$, so $\mu(g(D)) = 0$. Since this map is monotone on closed set D , it can be continued to a homeomorphism of the entire interval $[0, 1]$.

Problem 3. a) It's obvious that operator T preserves the norm. So, it's bounded, and, therefore, continuous.

b) Let's denote by x_k the k -th component of x : $x_k = (x, e_k)$. Then $(T^n x)_k = x_{\pi^n(k)}$. Now for any k we have two possibilities: either $\pi^n(k) = k$ for some n , or $\pi^n(k) \rightarrow \infty$, as $n \rightarrow \infty$.

If $\pi^n(k) = k$ then

$$S_N(k) = \frac{1}{N} \sum_0^N (T^i x)_k \rightarrow A(k) = \frac{1}{n} \sum_0^n (T^i x)_k$$

when $N \rightarrow \infty$. Indeed, $S_N(k) = A(k) + \frac{1}{N} \sum_{[N/n]n}^N (T^i x)_k$ and $\frac{1}{N} \sum_{[N/n]n}^N (T^i x)_k \rightarrow 0$ when $N \rightarrow \infty$.

If $\pi^n(k) \rightarrow \infty$ when $n \rightarrow \infty$ then $S_N(k) \rightarrow A(k) = 0$. Indeed, $x_m < \varepsilon$ when $m > M$, so

$$S_N(k) = \frac{1}{N} \sum_0^{N_1} (T^i x)_k + \frac{1}{N} \sum_{N_1+1}^N (T^i x)_k,$$

where N_1 is such that $\pi^n(k) > M$ when $n > N_1$. Since N_1 doesn't depend on N the first term tends to zero. And the second term is less than ε . So, $S_N(k) \rightarrow 0$ when $N \rightarrow \infty$.

So, we see that the limit exists for any component. We need to show that corresponding vector converges in H . We can choose such M that $\sum_k |x_k|^2 < \varepsilon$ if $k > M$. Let $K \geq M$ be such number that for all $k > K$ we have $\pi^n(k) > M$ for any n . Then we have $\sum_{k \geq K} |S_N(k)|^2 < \varepsilon$, so $\sum_k |S_N(k) - A(k)|^2 < 2\varepsilon$ for any N . Also, we can find such N that $\sum_{k=1}^K |S_N(k) - A(k)| < \varepsilon$. So, we see that the averages $\frac{1}{N} \sum_0^{N_1} T^i x$ have a limit in H .

c) Denote $z_n = \frac{1}{n} \sum_0^{n-1} T^k x$. Then

$$Tz_n = \frac{n+1}{n} z_{n+1} - \frac{x}{n},$$

So, $Tz_n - z_{n+1} \rightarrow 0$, as $n \rightarrow \infty$. And since $z_n \rightarrow z$, we have $Tz = z$.

Problem 4. a) Let

$$f(x) = \frac{1}{x(1 - \log x)^2}.$$

Then

$$\int_0^1 |f(x)| dx = \frac{1}{1 - \log x} \Big|_0^1 = 1,$$

and $\int_0^1 |f(x)|^p dx = \infty$ for any $p > 1$.

b) Obviously, $f^{1/p}$, where f is from the previous part, works.

c) Yes, since you can consider $f\chi_{[0,1]} \in \mathbb{L}_1(\mathbb{R})$.

Problem 5.

We know that

$$s_n(f)(x_0) - f(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x_0 + z) - f(x_0)}{z} \frac{z}{2 \sin \frac{z}{2}} \sin \frac{2n+1}{2} z dz.$$

So, if $f'(x_0)$ exists, then the function $g(z) = \frac{f(x_0+z)-f(x_0)}{z} \frac{z}{2 \sin \frac{z}{2}}$ is integrable. Then, by Riemann lemma, we have

$$s_n(f)(x_0) - f(x_0) = \int_{-\pi}^{\pi} g(z) \sin \frac{2n+1}{2} z dz \rightarrow 0, \quad \text{when } n \rightarrow \infty.$$