ANTON GORODETSKI

253-37 Mathematics, Caltech, Pasadena CA 91125 *(626) 395-4350, (626) 403-0386; asgor@caltech.edu*
www.its.caltech.edu/~asgor

Research Statement

My research interests are mostly in the field of smooth dynamical systems, including

- partially hyperbolic and non-uniformly hyperbolic dynamics;
- nonlocal and homoclinic bifurcations;
- structure of Newhouse domains (in both dissipative and conservative setting);
- applications to problems in other fields of mathematics (including celestial mechanics and discrete Schrödinger equations).

Here is the list of main results:

- one parameter version of the conservative Newhouse phenomena (a version of *Conservative Palis' Conjecture*);
- Hausdorff dimension of oscillatory motions for the Sitnikov example and the Restricted Planar Circular 3 Body Problem is often maximal possible (counterpart of *Kolmogorov's conjecture*);
- finitude of attracting periodic orbits of a given cyclicity near a homoclinic contour for a typical (in a measure-theoretical sense) perturbation (weak form of *Palis' Conjecture on finitude of attractors*);
- construction of non-hyperbolic measures in partially hyperbolic dynamical systems;
- Hölder property of central foliations of partially hyperbolic maps close to direct products;
- description of wild dynamical properties of skew products that appear in partially hyperbolic dynamics and via some nonlocal bifurcations.

Below I describe these and other results in more details. The last paragraph of each section is devoted to future research.

1. Finitude of attractors.

R.Thom conjectured that a typical diffeomorphism has at most a finite number of attractors [T]. S.Newhouse proved that in the space of C^r-smooth diffeomorphisms $\text{Diff}^r(M)$, $r \geq 2$, of a compact surface M there is an open set U such that a Baire generic diffeomorphism $f \in U$ has infinitely many coexisting sinks [N2], [N3]. Therefore, topologically generic dynamical system may have infinite number of attractors. It is not known if a prevalent (generic in a measure-theoretical sense) dynamical system can or can not have this property. Palis' Conjecture on finitude of attractors claims that, roughly speaking, with probability one a dynamical system has only finite number of attractors.

Infinite number of sinks for two dimensional diffeomorphisms appear near homoclinic tangencies. Let $f \in \text{Diff}^r(M)$ be a diffeomorphism with a homoclinic tangency. Let V be a small neighborhood of a homoclinic contour. A sink is *V-localized* if its trajectory is inside of V. It has *a complexity s* if it turns around V exactly s times. In our joint work with V.Kaloshin [GK1] we show that *for any integer s a prevalent (typical in measure-theoretical sense) unfolding of the diffeomorphism f has only finite number of V-localized sinks of complexity s.* Similar result holds if we consider sinks of arbitrary complexity, but with periods that are much larger than complexity. This statement can be considered as a far reaching generalization of [LY], where sinks of complexity one were treated.

The approach we use involves Newton Interpolation Polynomials and Discretization method, developed in [K1], [KH]. For a separate decription of the method see [GHK]. It seems reasonable to hope that these techniques can be applied to some of the most difficult problems of smooth dynamical systems, such as, for example, behavior of Henon and Henon-like maps, see [H], [BC], [MV], [WY].

2. Conservative Newhouse phenomena.

S. Newhouse [N1] constructed open sets (nowadays called *Newhouse domains*) of maps where diffeomorphisms with tangencies between stable and unstable manifolds (*homoclinic tangencies*) are dense. Moreover, in these open sets there are residual subsets of maps with infinitely many attracting periodic orbits (this provides a counterexample to Thom's conjecture, see previous section). Later C.Robinson [R] noticed that this result can be formulated in terms of generic one parameter unfolding of a homoclinic tangency.

P. Duarte [D] showed that in area preserving case existence of homoclinic tangencies also lead to similar phenomena, where sinks are replaced by elliptic islands.

In my part of [GK2] I prove a stronger one parameter version of the Duarte's result. Namely, consider a generic unfolding of a quadratic homoclinic tangency associated with a saddle P_0 of an area preserving map f_0. *There is an open set U in the space of parameters such that for every $\mu \in U$ the map f_μ has a hyperbolic set with persistent homoclinic tangencies. Moreover, for every μ from some residual subset of U the map f_μ has an invariant transitive closed set H_μ such that*

- *the set H_μ is accumulated by f_μ's elliptic points,*
- $\dim_H H_\mu = 2$,
- $\dim_H \{x \in H_\mu | \, P_\mu \in \omega(x) \cap \alpha(x)\} = 2$, *where P_μ is the unique fixed point near P_0.*

Our interest to this topic initially was motivated by applications to celestial mechanics described in the next section.

What one can say about the Lebesque measure of H_μ or $\{x \in H_\mu | \, P_\mu \in \omega(x) \cap \alpha(x)\}$? This question is related to Kolmogorov's Conjecture on the measure of oscillatory motions in the three body problem. Also it is motivated by similar problems regarding the standard map.

Another challenge is to find an analog of these results in higher dimension (in particular, to construct Newhouse domains in dimension four). Even in dissipative case such generalization is quite far from being trivial, see [PV].

3. Hausdorff dimension of oscillatory motions.

The modern theory of Dynamical Systems is in major part an offspring of celestial mechanics. Poincare proved non-integrability of the three body problem when he discovered the homoclinic picture. Alexeev [A] explained the existence of the oscillatory motions in Sitnikov model (one of the restricted versions of the three body problem) using methods of hyperbolic dynamics. A motion of the 3 body problem is called oscillatory if limsup of maximal distance among the bodies is infinity as time tends to infinity and liminf is finite. Kolmogorov conjectured that the set of oscillatory motions has zero measure.

In our joint work with V.Kaloshin [GK2] we show that in many cases the set of oscillatory motions in the 3 body problem has maximal Hausdorff dimension. Proof relies on investigation of objects exposed in the previous section, such as area-preserving Henon family, persistent homoclinic tangency, splitting of separatrices. Namely, consider the Sitnikov problem. It is a special case of the restricted three body problem where the two primaries with equal masses are moving in an elliptic orbits of the two body problem, and the infinitesimal mass is moving on the straight line orthogonal to the plane of motion of the primaries which passes through the center of mass. Eccentricity e_0 of orbits of primaries is a parameter. After some change of coordinates

(McGehee transformation) the infinity can be considered as a degenerate saddle with smooth invariant manifolds that correspond to parabolic motions (the orbit tends to infinity with zero limit velocity). Stable and unstable manifolds coincide in the case of circular ($e_0 = 0$) Sitnikov problem. Dankowicz and Holmes [DH] showed that for non-zero eccentricity invariant manifolds have a point of transverse intersection. This leads to the existence of homoclinic tangencies and appearance of all the phenomenon that can be encountered in the conservative homoclinic bifurcations, as described in the previous section. In particular, existence of hyperbolic sets of large Hausdorff dimension implies that *there is an open set U, $0 \in \overline{U}$, in the space of parameters such that for parameters from some residual subset of U the set of oscillatory orbits in the Sitnikov problem has full Hausdorff dimension. Similar statement holds for the planar circular restricted three body problem.* The existence of transversal homoclinic points in the latter case was established in [LS], [X].

In our attempts to generalize this result to the versions of the three body problem with smaller groups of symmetries we encountered serious technical difficulties of pure dynamical nature. Namely, conservative homoclinic bifurcations are much harder to investigate in higher dimensions due to the fact that there are no smooth stable and unstable foliations near hyperbolic sets in this case. This motivates an extensive family of problems to be attacked in the near future.

4. Central foliations of partially hyperbolic maps.

Two mechanisms of structural instability in smooth dynamical systems are known: partial hyperbolicity and homoclinic tangencies. It was conjectured by J. Palis [P] that at least one of these phenomena is always present in essentially structurally unstable systems.

The simplest way to construct a partially hyperbolic map is to consider a direct product $\mathfrak{F}$ of a map $S : N \to N$ exhibiting a uniformly hyperbolic set Λ by identity map of a compact manifold $\mathrm{id} : M \to M$, $\mathfrak{F} = S \times \mathrm{id}_M$. It is well known (see [HPS]) that invariant set $\Lambda \times M$ does not disappear after small perturbations of the map $\mathfrak{F}$. In [G2] I proved that *a small C^r-perturbation of the map $\mathfrak{F}$ has an invariant partially hyperbolic set which central leaves depend Hölder continuously on the point on transversal in C^{r-1}-metric.* This is a substantial generalization of a result by V. Nitica and A. Török [NT], where the partial case ($M = \mathbb{T}^k$, $S : \mathbb{T}^l \to \mathbb{T}^l$ is Anosov) was considered.

The proof uses the compactness of the central leaves. Is it possible to generalize this result to the case of non-compact leaves? It is an open and important question (see [PSW]).

5. Non-removable zero Lyapunov exponents.

To what extent is the behavior of a generic dynamical system hyperbolic? A number of problems in modern theory of smooth dynamical systems can be viewed as some forms of this question. It was shown in the 1960s that uniformly hyperbolic systems (Anosov diffeomorphisms, Axiom A) are not dense in the space of dynamical systems [AS]. This forced weakening the notion of hyperbolicity and gave rise to notions of partial and nonuniform hyperbolicity. Some systems of these types are studied by famous Pesin theory [Pe]. In Pesin theory, hyperbolic behavior is characterized by nonzero Lyapunov exponents for some invariant measure. The most natural case is that of a system with a smooth invariant measure. This case was studied in various aspects, for example, in [B], [BB], [DP]. However, the question about Lyapunov exponents can also be considered for maps that do not a priori have a natural invariant measure.

In [GI1], [GI2] we investigate the properties of skew products generated by partially hyperbolic maps considered in the previous section. Namely, the map in the base is a hyperbolic map (one can take a topological Bernoulli shift as a model of a hyperbolic map), and a map on a fiber M depends Hölder continuously on a point in base. In this way *for an open set of diffeomorphisms we construct an orbit with a zero Lyapunov exponent*. Moreover, in [GIKN] we show how not just an orbit, but also *an invariant ergodic non-hyperbolic (with a zero Lyapunov exponent) measure* can be constructed for a class of skew products. Using the result above related to Hölder property of central leaves my co-authors V.Kleptsyn and M.Nalsky [KN] showed that the same conclusion works for an open set of diffeomorphisms, not just skew products.

It is reasonable to conjecture that *uniformly hyperbolic maps together with diffeomorphisms exhibiting a non-hyperbolic ergodic invariant measure form an open and dense subset in the space of smooth dynamical systems.* In [DG] we prove that *this is true for robustly hyperbolic diffeomorphisms.* The main ingredient of the proof is a partially hyperbolic structure of robustly transitive diffeomorphisms [BDPR]. It would be very interesting to understand other mechanisms that create non-hyperbolic invariant measures.

6. Nonlocal bifurcations.

In our joint work with Yu.Ilyashenko [GI2] we described a bifurcation of a vector field with a saddlenode cycle exhibiting several invariant homoclinic tori. In particular, we show that the maps described in Sections 4 and 5 above appear as a Poincaré map for many (positive density at the point of bifurcation) parameter values.

7. Discrete Scrödinger equations.

Currently I am very interested in dynamical approach to discrete Schrödinger equations. In particular, I would like to apply the results on non-uniformly hyperbolic linear cocycles (see [Y]) to the discrete Scrödinger equation with smooth quasiperiodic potential. Also I am going to study the relation between spectral properties of discrete Schrödinger operators with Fibonacci potential and dynamical properties of the so called trace map (see [C]). The latter problem was suggested to me by D.Damanik.

References

[DG] Diaz L., Gorodetski A., Non hyperbolic invariant measures produced by heteroclinic cycles, in preparation.

[G1] Gorodetski A., Hierarchy of attractors for Axiom A diffeomorphisms, *Vestnik MSU*, no.1 (1996), pp. 84-86.

[G2] Gorodetski A., Regularity of central leaves of partially hyperbolic sets and applications, *Izvestiya RAN*, vol.70 (2006), no.6, pp. 52-78.

[GHK] Gorodetski A., Hunt B., Kaloshin V., Newton interpolation polynomials, discretization method, and certain prevalent properties in dynamical systems, Vol.2, *Proceedings of ICM 2006, Madrid, Spain, European Math Society*, 2006, pages 27–55.

[GI1] Gorodetski A., Ilyashenko Yu.S., Minimal and strange attractors, *International Journal of Bifurcation and Chaos*, vol.6, no.6 (1996), pp. 1177-1183.

[GI2] Gorodetski A., Ilyashenko Yu.S., Some new robust properties of invariant sets and attractors of dynamical systems, *Funct. Analysis and Applications*, no.2, vol.33, 1999, pp. 16-30.

[GI3] Gorodetski A., Ilyashenko Yu.S., Some properties of skew products over the horseshoe and solenoid, *Proc.of Steklov Institute*, vol.231 (2000), pp. 96-118.

[GIKN] Gorodetski A., Ilyashenko Yu.S., Kleptsyn V., Nalskiy M., Nonremovability of zero Lyapunov exponents, *Funct. Analysis and Applications*, vol.39 (2005), no. 1, pp. 21-30.

[GK1] Gorodetski A., Kaloshin V., How often surface diffeomorphisms exhibit infinite number of sinks, *Advances in Mathematics*, 208 (2007), pp. 710-797.

[GK2] Gorodetski A., Kaloshin V., Hausdorff dimension of oscillatory motions in the restricted planar circular three body problem and in Sitnikov problem, in preparation.

[A] Alexeyev V., Sur l'allure finale du mouvement dans le problème des trois corps. *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Gauthier-Villars, Paris, 1971, Tome 2, pp. 893-907.

[AS] Abraham R. and Smale S., Nongenericity of Ω-stability. *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, Amer. Math. Soc., Providence, R.I., 1970, pp. 5-8.

[B] Bochi J., Genericity of zero Lyapunov exponents, *Ergodic Theory Dynam. Systems*, 22 (2002), no. 6, pp. 1667-1696.

[BB] Baraviera A. and Bonatti C., Removing zero Lyapunov exponents, *Ergodic Theory Dynam. Systems*, 23 (2003), pp. 1655-1670.

[BC] Benedicks M., Carleson L., The dynamics of the Henon map. *Ann. of Math.* (2) 133 (1991), no. 1, pp. 73–169.

[BDPR] Bonatti C., Díaz L., Pujals E., Rocha J., Robustly transitive sets and heterodimensional cycles. *Geometric methods in dynamics. I. Astérisque* No. 286 (2003), xix, pp. 187-222.

[C] Casdagli M., Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation, *Comm. Math. Phys.*, vol.107 (1986), no. 2, pp. 295-318.

[DH] Dankowicz H., Holmes P., The existence of transverse homoclinic points in the Sitnikov problem. *J. Differential Equations*, vol. 116 (1995), no. 2, pp. 468-483.

[DP] Dolgopyat D. and Pesin Ya., Every compact manifold carries a completely hyperbolic diffeomorphism, *Ergodic Theory Dynam. Systems*, 22 (2002), no. 2, pp. 409-435.

[D] Duarte P., Abundance of elliptic isles at conservative bifurcations, *Dynamics and Stability of Systems*, vol. 14, no. 4, (1999), pp. 339-356.

[Ge] Gelfreich V., Splitting of a small separatrix loop near the saddle-center bifurcation in area-preserving maps, *Physica D* 136 (2000), no. 3-4, pp. 266-279.

[GS] Gelfreich V., Sauzin D., Borel summation amd splitting of separatrices for the Henon map, *Ann.Inst.Fourier, Grenoble* 51, 2 (2001), pp. 513-567.

[H] Henon M., A two-dimensional mapping with a strange attractor, *Comm. Math. Phys.*, 50 (1976), no. 1, pp. 69–77.

[HPS] Hirsh M., Pugh C., Shub M., Invariant manifolds, *Lecture Notes in Math.* (1977), vol.583.

[HSY] Hunt B., Sauer T., Yorke J., Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces, *Bull. Amer. Math. Soc. (N.S.)* 27 (1992), no. 2, pp. 217-238.

[KH] Kaloshin V., Hunt B., A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms. I. to appear in *Annals of Mathematics*, 94 pages.

[K] Kaloshin V., Some prevalent properties of smooth dynamical systems, *Proc. Steklov Inst. Math.* 1996, no. 2 (213), pp. 115-140.

[K1] Kaloshin V., Ph.D. thesis, Princeton University, 2001.

[KN] Kleptsyn V., Nalskiy M., Stability of existence of non-hyperbolic measures for C^1-diffeomorphisms, to appear in *Funct. Analysis and Applications*.

[LS] Llibre J., Simo C., Oscillatory solutions in the planar restricted three-body problem,*Math. Ann.* **248**, (1980), pp. 153-184.

[LY] Tedeschini-Lalli L., Yorke J. How often do simple dynamical processes have infinitely many coexisting sinks? *Comm. Math. Phys.* **106**, (1986), no. 4, pp. 635-657.

[MM] McCluskey, H., Manning, A., Hausdorff dimension for horseshoes, *Ergodic Theory Dynam. Systems* 3 (1983), no. 2, pp. 251-260.

[MR] Mora, L., Romero, R., Moser's invariant curves and homoclinic bifurcations, *Dynamic Systems and Applications*, (1997), vol. 6, pp. 29-42.

[MV] Mora L., Viana M., Abundance of strange attractors. *Acta Math.* 171 (1993), no. 1, pp. 1-71.

[N1] Newhouse S., Non-density of Axiom A(a) on S^2, Proc. A.M.S. Symp. Pure Math., **14**, (1970), 191–202;

[N2] Newhouse S., Diffeomorphisms with infinitely many sinks. *Topology* **13**, (1974), pp. 9-18.

[N3] Newhouse S., The abundance of wild hyperbolic sets and nonsmooth stable sets of diffeomorphisms, *Publ. Math. I.H.E.S.*, **50**, (1979), pp. 101-151.

[NT] Nitica V., Török A., Cohomology of dynamical systems and rigidity of partially hyperbolic actions of higher-rank lattices, *Duke Math. J.*, 1995, vol.79, no.3.

[P] Palis J., A global view of dynamics and a conjecture on the denseness of finitude of attractors. Géométrie complexe et systèmes dynamiques (Orsay, 1995). *Astérisque* No. 261 (2000), xiii–xiv, pp. 335-347.

[Pe] Pesin Ya., Characteristic Lyapunov exponents and smooth ergodic theory, *Usp. Mat. Nauk*, 32 (1977), No. 4 (196), pp. 55-112.

[PT] Palis J., Takens F., Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. *Cambridge University Press*, 1993.

[PV] Palis J., Viana M., High dimension diffeomorphisms displaying infinitely many periodic attractors. *Ann. of Math. (2)* 140 (1994), no. 1, pp. 207-250.

[PY] Palis J., Yoccoz J.-C., Fers à cheval non uniformément hyperboliques engendrés par une bifurcation homocline et densité nulle des attracteurs. (French) [Non-uniformly hyperbolic horseshoes generated by homoclinic bifurcations and zero density of attractors] *C. R. Acad. Sci. Paris Sér. I Math.* 333 (2001), no. 9, pp. 867-871.

[PSW] Pugh C., Shub M., Wilkinson A., Hölder foliations, *Duke Math. J.*, 1997, vol.86, no.3, pp. 517-546.

[R] Robinson C., Bifurcations to infinitely many sinks. Comm. Math. Phys. **90**, (1986), no.3, pp. 433-459.

[T] Thom R., Stabilite structurelle et morphogenese. (French) Essai d'une theorie generale des modules. *Mathematical Physics Monograph Series. W. A. Benjamin, Inc., Reading, Mass.*, 1972.

[WY] Wang Q., Young L.-S., Strange attractors with one direction of instability. *Comm. Math. Phys.* 218 (2001), no. 1, pp. 1-97.

[X] Xia J., Melnikov method and transversal homoclinic points in the restricted three-body problem, *J. of Diff. Equations*, vol 96, 1 (1992), pp. 170-184.

[Y] Young L.-S., Lyapunov exponents for some quasi-periodic cocycles, *Ergodic Theory and Dynamical Systems*, vol.17 (1997), no. 2, pp. 483-504.