

A compact fourth-order-accurate embedded boundary method for the wave equation

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Abstract

A fourth-order-accurate embedded boundary method for the wave equation with Dirichlet or Neumann boundary conditions is presented. The method uses a compact finite-difference approximation for spatial derivatives and a modified equation approach to achieve fourth-order-accuracy in time.

1 Introduction

Finite difference discretizations with the boundary embedded in a Cartesian grid provide an alternative to body-fitted overset-grid methods, finite-element and finite-volume methods for handling complex geometries. Embedded boundary methods, being based on structured Cartesian grids, are highly efficient both in terms of operations and required memory per degree of freedom. They are also well suited for massively parallel computers as they are easy to load balance and scale to thousands of cores. Another advantage is that the geometry can be represented in a lower dimension and only local information such as the location and the normal of the boundary are required to enforce the boundary conditions, no costly (parallel) grid generation is needed.

Here we focus on the wave equation and outline a high-order-accurate embedded boundary method based on compact finite-difference approximations of spatial derivatives. We describe two approaches, related to the second-order-accurate methods described in [3] and [4], to impose the boundary conditions. However, unlike [3] and [4] where values are assigned by *extrapolation* in ghost points *outside* the boundary, we impose the boundary conditions by assigning values via *interpolation* to “ghost points” just inside the boundary. Our new approach is more accurate and removes the small cell stiffness problem for the Dirichlet problem as the boundary is always well separated from the point “inside” the ghost point. For the Neumann problem we mitigate the small cell problem by adding a fifth order accurate regularizing term to the interpolant before taking the normal derivative of the interpolant.

An advantage with our method compared to the

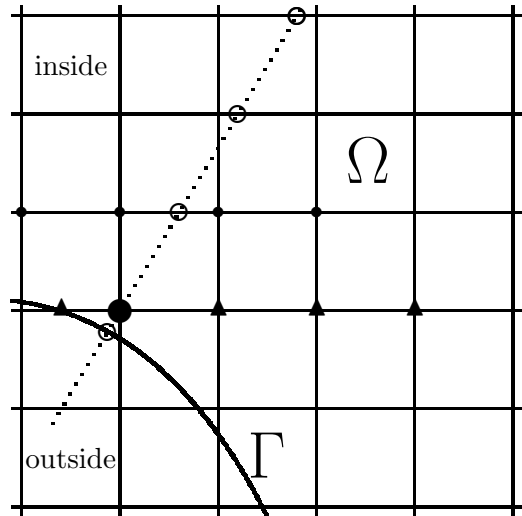


Figure 1: The dimension by dimension approach can be used for Dirichlet boundary conditions, interpolation in the normal direction can be used for both Dirichlet and Neumann boundary conditions.

Note that we place the ghost point *inside*.

unconditionally-stable fc-ad solver for the Dirichlet problem in [6] is that both Neumann and Dirichlet boundary conditions can be handled within the same framework. Also, our method is easily extended to other wave equations and first order systems unlike other high-order-accurate embedded boundary methods, as [5] and [8], that are explicitly designed for the wave equation.

2 The method

In this section we briefly describe the different building blocks of the method. We consider the wave equation $u_{tt} = \Delta u + f$, $(x, y) \in \Omega$ with Dirichlet or Neumann boundary conditions at the boundary Γ .

2.1 Spatial discretization

We approximate u_{xx} and u_{yy} dimension by dimension using the classical Pade scheme [1]. For example, let v be the grid function then $v_{xx} \approx u_{xx}$ is found by

solving the tri-diagonal system $Au_{xx} = b(v)$, where:

$$\begin{aligned} A_{0,0} &= 1, & A_{0,1} &= 11, & A_{N,N} &= 1, & A_{N,N-1} &= 11, \\ A_{\nu,\nu} &= 1, & A_{\nu,\nu-1} &= A_{\nu,\nu+1} &= 0.1, & \nu &= 1, \dots, N-1, \\ b_0 &= h^{-2}(13v_0 - 27v_1 + 15v_2 - v_3), \\ b_\nu &= 1.2h^{-2}(v_{\nu+1} - 2v_\nu + v_{\nu-1}), & \nu &= 1, \dots, N-1, \\ b_N &= h^{-2}(13v_N - 27v_{N-1} + 15v_{N-2} - v_{N-3}). \end{aligned}$$

2.2 Boundary conditions

Suppose v_0 is to be assigned at the large solid circle in Fig. 1. For the Dirichlet problem we can either construct an interpolant from the values in the points marked by triangles (one being the boundary condition) or from the empty circles along the normal. The values in the empty circles are found from interpolation from values at the small solid circles. For Neumann boundary conditions we must use the interpolant along the normal.

2.3 Temporal discretization

To achieve fourth-order-accuracy in time we follow [2] and expand $u(x, y, t)$ in t and replace temporal- with spatial-derivatives (approximated by repeated use of the above method) and advance the solution:

$$\begin{aligned} v^{n+1} &= 2v^n - v^{n-1} + k^2(\Delta v^n + f) \\ &+ \frac{k^4}{12}(\Delta(\Delta v^n) + \Delta f^n + f_{tt}^n). \end{aligned} \quad (1)$$

2.4 Stability

As with most embedded boundary methods we need to add a small damping term to the equations to get a stable method. To retain the fourth-order-accuracy we add $\gamma k^2 h^5 (v_{xxxxxx}^n + v_{xxxxxx}^n)$ (approximated as in [7]) to the right hand side in equation (1). From numerous computations we have determined that $\gamma = 0.008$ is sufficient to ensure long-time stability.

3 Results

In Figure 2 a sample computation is presented for a problem with homogeneous Dirichlet boundary conditions and initial data consisting of a Gaussian pulse at rest at the center of the domain.

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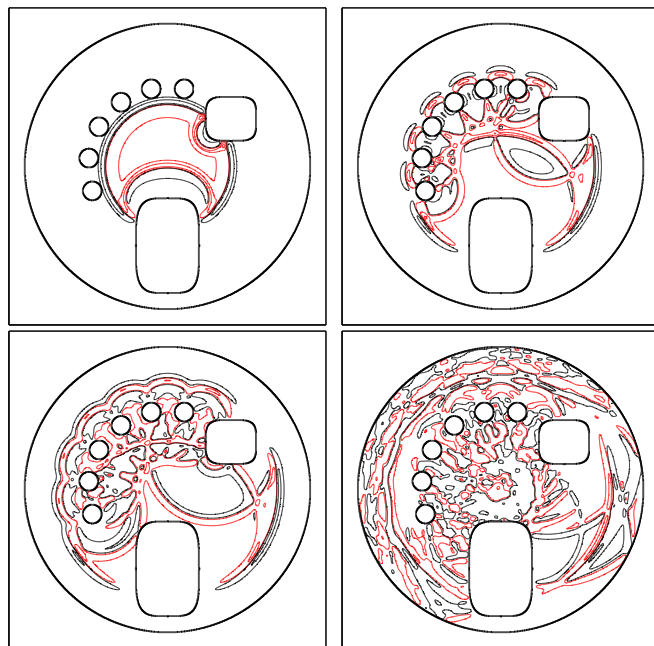


Figure 2: A pulse inside a circle with some solid obstacles at time 0.2, 0.3, 0.35 and 0.55.

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