Discretely Nonreflecting Boundary Conditions for Higher Order Centered Schemes for Wave Equations

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Abstract. Using the framework introduced by Rawley and Colonius [2] we construct a nonreflecting boundary condition for the one-way wave equation spatially discretized with a fourth order centered difference scheme. The boundary condition, which can be extended to arbitrary order accuracy, is shown to be well posed. Numerical simulations have been performed showing promising results.

1 Introduction

The goal of this paper is to present the construction of a *discretely* nonreflecting boundary condition for the one-way wave equation for the standard five-point central difference approximation of $\frac{\mathrm{d}u}{\mathrm{d}x}$. In a recent paper [2] Rawley and Colonius have provided an excellent framework for such construction. They use a technique where they decompose the solution into physical and spurious waves. The spurious waves have in general not been taken into account in continuous analysis of nonreflecting boundary conditions (e.g.[4]), which is essential for a boundary condition to be *discretely* nonreflecting.

In [2] Rawley and Colonius construct boundary conditions for the Padé three-point central difference scheme to illustrate their analysis.

Widening the scheme from three to five points allows for an explicit fourth order scheme. The Padé scheme, used in [2], is a fourth order implicit scheme. However when using a five-point scheme the wellposedness of the boundary condition is no longer trivial and needs to be treated carefully.

2 The one way Wave Equation $u_t + u_x = 0$

Consider the scalar one way wave equation

$$u_t + u_x = 0, \tag{1}$$

which admits solutions of the form

$$u(x,t) = e^{i(kx-\omega t)}. \tag{2}$$

It is well known that the discrete dispersion relation one obtains when a PDE is approximated by a difference scheme is different from the continuous dispersion relation. For example when (1) is spatially discretized with the second order centered difference scheme ($(u_x)_j \approx (u_{j+1} - u_{j-1})/(2h)$) on a regular grid with spacing h the discrete dispersion relation becomes $\omega = 2/h \sin kh$, which can be compared to the continuous dispersion relation $\omega = k$.

For each frequency ω there are two complex values of k that satisfy the discrete dispersion relation. The smaller one corresponds to a physical solution and the larger one corresponds to a spurious solution. The spurious solutions for this particular scheme are poorly resolved waves with a negative group velocity, i.e. they travel the wrong direction.

For the fourth order scheme (3) there exist four values of k that satisfies it's dispersion relation. Of these four, two correspond to wave solutions and two to damped wave solutions. The two damped solutions will not be important for the nonreflectivity of the boundary condition. However, they need to be taken into account when considering the wellposedness of the boundary condition.

3 Separation of Spurious and Physical Modes

Introducing a regular grid in x, with mesh spacing h, and letting u_k denote the approximation of $u(x = kh)$. We are interested of the dispersion relation of the fourth order centered difference stencil

$$(u_x)_k \approx \frac{1}{12h}(-u_{k+2} + 8u_{k+1} - 8u_{k-1} + u_{k-2}), \tag{3}$$

which applied to (1) yields

$$(u_t)_k = \frac{1}{12h}(u_{k+2} - 8u_{k+1} + 8u_{k-1} - u_{k-2}). \tag{4}$$

By introducing a normal mode solution

$$u_k(t) = \hat{u}e^{i\omega t}\kappa^k,$$

with the property

$$u_{k+1} = \kappa u_k.$$

We obtain the characteristic equation

$$N(i\omega, \kappa) \equiv \left[\kappa^4 - 8\kappa^3 - i12\omega h\kappa^2 + 8\kappa - 1\right] = 0,$$

which has four nontrivial solutions

$$\kappa^+ = 2 + \frac{1}{2}\xi - \frac{1}{2}\sqrt{\gamma + \theta}, \quad \kappa^- = 2 - \tfrac{1}{2}\xi - \tfrac{1}{2}\sqrt{\gamma - \theta},$$
$$\kappa^* = 2 + \frac{1}{2}\xi + \frac{1}{2}\sqrt{\gamma + \theta}, \quad \kappa^\div = 2 - \tfrac{1}{2}\xi + \tfrac{1}{2}\sqrt{\gamma - \theta},$$

where

$$\begin{gathered}\eta = (-18I\phi + 8I\phi^3 + \sqrt{-125 - 24\phi^2 + 48\phi^4})^{1/3},\\ \xi = \sqrt{16 + 8I\phi + (180 - 144\phi^2)/(18\eta) + 2\eta},\\ \gamma = 32 + 16I\phi - (180 - 144\phi^2)/(18\eta) - 2\eta,\\ \theta = (448 + 384I\phi)/(4\xi), \quad \phi = \omega h, \quad I = \sqrt{-1}.\end{gathered}$$

In order to distinguish the physical part of the solution from the spurious we write the solution u_k at a grid-point k as

$$u_k = u_k^+ + u_k^- + u_k^* + u_k^\div,$$

where $u_k^@$ are normal modes that satisfy

$$N(i\omega, \kappa^@)u^@ = 0, \quad @ = +, -, *, \div.$$

The solutions u^+ and u^- denote the physical and the spurious wave while u^* and $u^\div$ are damped waves that decay from the right and left boundary respectively.

4 Approximate Nonreflecting Boundary Conditions

We now seek approximate nonreflecting boundary conditions at the left ($k = 0$) and right ($k = N$) boundary. Following the ideas of Colonius [1] and Colonius and Rowley [2] we seek approximate boundary conditions of the form

$$\frac{du_0}{dt} = \frac{1}{c_1 h}\sum_{k=0}^{N_d} d_k u_k, \quad \frac{du_N}{dt} = \frac{1}{a_1 h}\sum_{k=0}^{N_d} b_k u_{N-k}, \tag{5}$$

where a_1, b_1, c_1, d_k, ($k = 0, \ldots, N_d$) are coefficients to be determined. Consider first the left boundary.

Taking the Fourier transform in time and splitting u into its modes (5) becomes

$$\begin{gathered}ic_1\omega h(u_0^+ + u_0^- + u_0^* + u_0^\div) = \sum_{k=0}^{N_d} d_k(u_k^+ + u_k^- + u_k^* + u_k^\div)\\ \iff ic_1\omega h u_0^+ - \sum_{k=0}^{N_d} d_k(\kappa^+)^k u_0^+ =\\ -ic_1\omega h(u_0^- + u_0^* + u_0^\div) + \sum_{k=0}^{N_d} d_k\left[(\kappa^-)^k u_0^- + (\kappa^*)^k u_0^* + (\kappa^\div)^k u_0^\div\right].\end{gathered} \tag{6}$$

With $\phi = \omega h$ (6) becomes

$$d^+(\phi)u_0^+ = d^-(\phi)u_0^- + d^*(\phi)u_0^* + d^\div(\phi)u_0^\div,$$

where

$$d^+(\phi) = c_1 i\phi - \sum_{k=0}^{N_d} d_k(\kappa^+)^k,$$

$$d^@(\phi) = -c_1 i\phi + \sum_{k=0}^{N_d} d_k(\kappa^@)^k, \quad @ = \{-, *, \div\}.$$

The exact nonreflecting boundary condition is $u_0^+ = 0$ would be satisfied if

$$d^@(\phi) = 0, \quad @ = \{-, *, \div\}.$$

It is not possible to fulfill $d^@(\phi) = 0$ for all $@ = \{-, *, \div\}$. However this is not necessary since $u^\div$ will be eliminated later and u^* must be specified at the right boundary. Thus only $d^-(\phi) = 0$ remains which is equivalent with the condition obtained in [2].

Since $d^-(\phi)$ contains powers of κ^- up to order N_d it is natural to expand $\kappa^-(\phi)$ as a power series expansion around $\phi = 0$. The coefficients $c_1, d_0, \ldots$ are choose to minimize $d^-(\phi)$.

At the right boundary we use the same technique to find the boundary condition for u_N. Separating the spurious and physical modes of the ansatz (5) leads to the following expression

$$b^+(\phi)u_N^+ = b^-(\phi)u_N^- + b^*(\phi)u_N^* + b^\div(\phi)u_N^\div,$$

where

$$b^+(\phi) = ac_1 i\phi - \sum_{k=0}^{N_d} \frac{b_k}{(\kappa^+)^k},$$

$$b^@(\phi) = -a_1 i\phi + \sum_{k=0}^{N_d} \frac{b_k}{(\kappa^@)^k}, \quad @ = \{-, *, \div\}.$$

The nonreflecting boundary condition for u_N is $b^+(\phi) = 0$.

5 Wellposedness

When applying the scheme (3) to the gridpoint $k = 1$ we need information of both u_0 and u_{-1}. Above u_0 has been determined to minimize the reflection at $k = 0$. The relation determining u_{-1} must be chosen such that the boundary condition is well posed. The mathematicly correct boundary condition is $u(t, 0) = 0$ which together with (1) gives the relation $u(t, 0)_{xx} = 0$. Using a three point wide skew stencil for approximation of $(u_{xx})_{k=0}$ one obtains the relation

$$u_{-1} = 2u_0 - u_1. \tag{7}$$

We have found that (7) yields a well posed boundary condition for choices of N_d at least up to $N_d = 8$. For example consider $N_d = 1$ which gives the following relation for u_0

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = -\frac{5}{3h}(u_0 + u_1). \tag{8}$$

To investigate the wellposedness of the above boundary conditions for (4) we treat each boundary separately using Laplace transform technique (see e.g. [3]). We obtain the following eigenvalue problem

$$\tilde{s}\psi_j = \frac{1}{12h}(\psi_{j+2} - 8\psi_{j+1} + 8\psi_{j-1} - \psi_{j-2}), \quad \tilde{s} = sh \tag{9}$$

There are two roots to the characteristic equation associated with (9), κ_ν, $\nu = 1, 2$, with $|\kappa_\nu| \leq 1$ for Re $\tilde{s} > 0$. The general solution of (9) with $\|\psi\|_h < \infty$ has the form

$$\phi_j = \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^j.$$

Using the boundary conditions (7) and (8) we obtain the following equations for determination of σ_1 and σ_2

$$\begin{aligned} \frac{\sigma_1}{\kappa_1} + \frac{\sigma_2}{\kappa_2} &= 2\sigma_1 + 2\sigma_2 - \kappa_1\sigma_1 - \kappa_2\sigma_2, \\ \tilde{s}(\sigma_1 + \sigma_2) &= \frac{5}{3}\left(-\sigma_1 - \sigma_2 - \kappa_1\sigma_1 - \kappa_2\sigma_2\right) \end{aligned}$$

which only admit the trivial solution $\sigma_1 = \sigma_2 = 0$, hence the boundary conditions are well posed.

For the right boundary $D_- u_{N-1}$ can be shown to yield a well posed boundary condition.

6 Numerical Experiments

We have made numerical simulations that show that our boundary condition is performing significantly better than characteristic bounary conditions. We especially see an improvement of the absorption of spurious waves.

References

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