Construction of stable PMLs for general 2×2 symmetric hyperbolic systems

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Abstract. The perfectly matched layer (PML) has emerged as an important tool for accurately solving certain hyperbolic systems on unbounded domains. An open issue is whether stable PMLs can be constructed in general. In this work we consider the specialization of our general PML formulation to 2×2 symmetric hyperbolic systems in $2 + 1$ dimensions. We show how to choose the layer parameters as functions of the coefficient matrices to guarantee stability.

1. Introduction

The perfectly matched layer (PML) model was introduced by Berenger [5] in the context of computational electromagnetics. In the Berenger approach the additional degrees of freedom needed for the perfect matching are obtained by splitting one of the physical fields, and his PML is therefore often referred to as the split-field model.

In [6] Collino and Tsogka showed how to use the split-field approach for the construction of a PML for a general hyperbolic system. The perfect matching property follows directly from the construction in [6], however there is no guarantee that the solution is damped inside the layer. Indeed this has been manifested by the presence of exponentially growing solutions in numerical simulations where the split-field PML has been used, see e.g [11], [2]. Even the original PML for Maxwell's equations supports linearly growing modes [3].

Different methods have been devised to remove growing modes, e.g. filtering, introduction of a complex frequency shift, and the addition of stabilizing parameters destroying the perfect matching. However, to our knowledge there exists no method

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to remove growing modes which is always applicable. Thus the theoretical question of the existence of a stable PML for general hyperbolic systems remains open.

The existence of growing solutions in split-field PML models has been studied in detail by Bécache, Fauqueux and Joly in [2]. They derive a necessary condition for the stability of such models for strongly hyperbolic systems which is interpreted in terms of the dispersion relation and the group velocity of the system. The fact that the group velocity condition is not sufficient is illustrated by numerical simulations of a split-field model for orthotropic elastic waves. It is somewhat surprising that the group velocity condition is only necessary and it may indicate that a simple necessary and sufficient condition will be difficult to obtain using the standard mathematical tools (Fourier and Laplace transforms). In this paper we focus on understanding the properties of another PML model for hyperbolic systems suggested in [9]. It is based on modification of the *modal solutions* of the original system such that solutions inside the layer are damped exponentially as they propagate. The analysis of modal solutions is important in the development and study of well-posedness of initial-boundary value problems for hyperbolic systems; see. e.g [13], [10]. Therefore we expect the analysis of PMLs constructed using the modal solution approach to be easier than for those constructed with the split-field approach. Another advantage, which allows the construction of a stable layer in more complex situations, is that the new model contains more free parameters.

In this paper we will analyze the PML from [9] for a general 2×2 symmetric hyperbolic system in $2 + 1$ dimensions. We note that many larger hyperbolic systems, such as the isotropic and anisotropic Maxwell equations, Euler's equation linearized around a uniform flow, the linearized shallow water equation, and the wave equation, can be characterized as a combination of the waves present in the model problem. The theory developed here can thus be applied to these equations to design stable PMLs.

As the proposed PML is matched by construction (see [8]), the main issue is how to choose parameters so that the solution is damped in the layer. Here we will show how the eigenvalues, obtained from the modal solution, can be used to determine the parameters. In the spirit of [2], we will relate the parameter choices to geometric conditions on the the slowness curve. The proofs of the exponential decay of the solution are somewhat lengthy and will be presented elsewhere.

2. The Model Problem and the PML

We consider the construction of a PML for the symmetric hyperbolic system

$$u_t = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}}_{A} u_x + \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}}_{B} u_y. \tag{1}$$

Here A and B are real matrices and we can choose $a_{12} = 0$ without loss of generality. Performing a Laplace transform in time and a Fourier transform in y we obtain the modal solutions of (1)

$$\hat{u} = e^{\lambda x}\hat{\phi}, \quad (-sI + \lambda A + ik_y B)\,\hat{\phi} = 0.$$

The PML is constructed by modifying the solutions in the layer so that $\Re\lambda \neq 0$ for all but finitely many $\Re s \geq 0$. If the eigenfunctions, $\hat{\phi}$, remain unchanged the layer will be perfectly matched.

Directly, the layer in the strip $(x, y) \in (0, L) \times (y_0, y_1)$, is constructed from the formal modal solution

$$\hat{u} = e^{\lambda x + \left(\frac{(\lambda - \gamma_1 + \gamma_0\alpha_0)}{s + \alpha_1 + \alpha_0} - \gamma_0\right)\int_0^x \sigma(z)dz}\hat{\phi}(s, ik_y),$$

where λ and $\hat{\phi}$ are the eigensolutions in the physical domain. It is straightforward to show that the modal solution corresponds to the system

$$\begin{aligned} u_t &= A(u_x + \sigma(\bar{\gamma}_0 u + w)) + Bu_y, \\ w_t + (\sigma + \bar{\alpha}_1 + \bar{\alpha}_0)w + u_x + \bar{\gamma}_0(\sigma + \bar{\alpha}_0)u - \bar{\gamma}_1 u &= 0, \end{aligned} \tag{2}$$

where we have introduced the auxiliary variable w. In this paper $\bar{\gamma}_1$ will be a differential operator of the form $g^y\partial_y$, where g^y is a real scalar parameter. Similarly $\bar{\alpha}_1$ will be of the form $a^y\partial_y$. Here $\bar{\gamma}_0$ and $\bar{\alpha}_0$ will be taken as a real scalars. Note that the layer will be perfectly matched for all $(\bar{\gamma}_1, \bar{\gamma}_0, \bar{\alpha}_1, \bar{\alpha}_0)$ so the main issue is to choose them so that the layer is damping.

Remark 1. The parameter $\bar{\alpha}_0$ is sometimes referred to as the complex frequency shift. It was introduced in the context of split-field PMLs for Maxwell's equations in order to prevent linear growth. Within the model (2) we will show that it can also be used to remove exponential growth in corners.

Remark 2. The model (2) can be formulated with even more parameters but here we omit them for the sake of clarity. For the most general formulation we refer to [8] where a proof of the matching property can also be found.

3. A Stable PML for the 2D Transport Equation

We first consider the case when $a_{12} = b_{12} = 0$, $a_{11} \neq 0$, $a_{22} \neq 0$, i.e. we have two decoupled transport equations. Starting with the construction of a PML for the first component of $u = (u^1, u^2)$

$$u^1_t = a_{11}u^1_x + b_{11}u^1_y, \tag{3}$$

we note that

$$\lambda = \frac{s - ib_{11}k_y}{a_{11}}.$$

To ensure damping in the propagation direction, the real part of the additional exponent in the layer should have the same sign as a_{11}. This can be accomplished by choosing $\alpha_1 = \alpha_0 = 0$, $\gamma_1 = -i\,k_y b_{11}/a_{11}$. Thus the modal solution in the layer becomes

$$\hat{u}^1 = \hat{\phi}^1(s, ik_y)e^{\lambda x + (\frac{1}{a_{11}} - \gamma_0)\int_0^x \sigma(z)dz}. \tag{4}$$

Inverting the transforms we obtain the solution in the layer

$$u(x,y,t) = \phi^1(t + x/a_{11}, y - x\, b_{11}/a_{11})\, e^{(\frac{1}{a_{11}} - \gamma_0)\int_0^x \sigma(z)dz},$$

which is clearly perfectly matched and exponentially decaying, both for left and rightgoing modes, so long as:

$$1 - a_{11}\gamma_0 > 0. \tag{5}$$

This is easily ensured, and we can obviously take $\gamma_0 = 0$. Note that one could also achieve damping in this case by only adding a γ_0 term of the appropriate sign.

For the second equation the analysis is the same but we choose $\gamma_1 = -i\, k_y b_{22}/a_{22}$. Written out and assuming $\gamma_0 = 0$ the system in the layer becomes

$$\begin{aligned} u_t &= A(u_x + \sigma w) + B u_y, \\ w_t + \sigma w + u_x - \bar{\Gamma}_1 u &= 0, \\ \bar{\Gamma}_1 &= \mathrm{diag}([-b_{11}/a_{11}\partial_y, -b_{22}/a_{22}\partial_y]). \end{aligned}$$

We note that a direct stability analysis in the time domain involves the computation of s from $\lambda = ik_x$ and ik_y. This leads to the quadratic equation:

$$s(s + \sigma(1 - a_{11}\gamma_0) - ia_{11}k_x - ib_{11}k_y) = 0$$

whose roots have nonnegative real part if (5) is satisfied.

3.1. PML for the 2D Transport Equation in a Corner

Following the analysis above we get (for the first component) the following layer in a corner,

$$\begin{aligned} u_t^1 &= a_{11}(u_x^1 + \sigma^x w^x) + b_{11}(u_y^1 + \sigma^y w^y), \\ w_t^x + (\bar{\alpha}_0^x + \sigma^x) w^x + u_x^1 + \frac{b_{11}}{a_{11}} u_y^1 &= 0, \\ w_t^y + (\bar{\alpha}_0^y + \sigma^y) w^y + \frac{a_{11}}{b_{11}} u_x^1 + u_y^1 &= 0. \end{aligned}$$

Here we have added the parameter $\bar{\alpha}_0$ in order to stabilize the corner layer. By the techniques described in [8] it can be shown that the corner layer will be stable if we choose $\bar{\alpha}_0^x = \nu^x\sigma^x$ and $\bar{\alpha}_0^y = \nu^y\sigma^y$ and require that $\nu^x\nu^y \geq 1$.

4. The General Case

Now we consider the case when $a_{12} = 0$, $b_{12} \neq 0$, $a_{11} \neq 0$, $a_{22} \neq 0$. Solving the equation $\det(-sI + \lambda A + ik_y B) = 0$ for λ we obtain

$$\begin{aligned} \lambda \;=\; & \frac{s(a_{11} + a_{22}) - ik_y(a_{11}b_{22} + b_{11}a_{22})}{2a_{11}a_{22}} \\ & \pm\sqrt{\left(\left(\frac{(a_{11} - a_{22})}{2a_{11}a_{22}}\right) s - \left(\frac{a_{11}b_{22} - b_{11}a_{22}}{2a_{11}a_{22}}\right) ik_y\right)^2 - \frac{b_{12}^2}{a_{11}a_{22}} k_y^2}. \end{aligned} \tag{6}$$

We consider two distinct cases: waves moving in each direction and waves all moving in the same direction. We note that the goal in each instance is to ensure that waves are damped in the direction of propagation as determined by the group velocity. By considering $\Re s > 0$, s sufficiently large, we note that the number of waves propagating in each direction is determined by the signs of the numbers a_{11} and a_{22}.

4.1. Waves Moving Both Ways

Without loss of generality we assume $a_{11} > 0$ and $a_{22} < 0$. Since we know that for $\Re s > 0$ there is one eigenvalue with positive real part and one with negative real part these must correspond to the signs of the real parts of the terms including the radicals in (6). Noting that $a_{11}a_{22} < 0$ we can define $\tilde{s}$, $\Re\tilde{s} > 0$ such that:

$$\lambda = -\mu\tilde{s} - \nu i k_y \pm \sqrt{\tilde{s}^2 + \frac{b_{12}^2}{|a_{11}a_{22}|}k_y^2}. \tag{7}$$

Precisely:

$$\tilde{s} = \frac{(a_{22} - a_{11})}{2a_{11}a_{22}}s + \frac{(a_{11}b_{22} - b_{11}a_{22})}{2a_{11}a_{22}}ik_y,$$

$$\mu = \frac{a_{11} + a_{22}}{a_{11} - a_{22}}, \quad \nu = \frac{b_{11} - b_{22}}{a_{11} - a_{22}}.$$

We now recall the basic fact used in the construction of stable PMLs for Maxwell's equations:

$$\Re\left(\frac{\sqrt{\tilde{s}^2 + \frac{b_{12}^2}{|a_{11}a_{22}|}k_y^2}}{\tilde{s} + \tilde{\alpha}_0}\right) > 0,$$

when $\Re\tilde{s} \geq 0$, $\tilde{s} \neq \pm i\frac{|b_{12}|}{\sqrt{|a_{11}a_{22}|}}k_y$. Thus a term of the form:

$$\exp\left(\frac{\lambda + \mu\tilde{s} + \nu i k_y}{\tilde{s} + \tilde{\alpha}_0}\int_0^x \sigma(z)dz\right),$$

will damp waves in the direction of propagation.

In terms of the layer parameters, after scaling the denominator so that it has the form $s + \alpha_0$ (note that the scaling can be absorbed into σ) we have:

$$\gamma_1 = \frac{b_{22} - b_{11}}{a_{11} - a_{22}}ik_y, \quad \gamma_0 = -\frac{a_{11} + a_{22}}{2|a_{11}a_{22}|}, \tag{8}$$

$$\alpha_1 = \frac{b_{11}a_{22} - b_{22}a_{11}}{a_{11} - a_{22}}ik_y, \quad \alpha_0 \geq 0. \tag{9}$$

Using the energy techniques in [8] we can prove the following lemma.

Lemma 4.1. *If $a_{11} > 0$ and $a_{22} < 0$ then the PML (2) is stable with the choices of parameters (8)-(9).*

4.2. Waves Moving One Way

Now the parameters a_{11} and a_{22} have the same sign, which corresponds to a single direction of propagation. Then the sign of the damping term is the same for each mode and it is sufficient to have γ_0 nonzero with the opposite sign as the a_{jj}. If instead we try to directly mimic the construction from the previous case we run into difficulties. For simplicity consider the case of $a_{11} > a_{22} > 0$, $b_{11} = b_{22}$, so that $\nu = 0$, and redefine $\tilde{s}$ by $-\tilde{s}$ so that we are still considering $\Re\tilde{s} > 0$. Then instead of (7) we have:

$$\lambda = \mu\tilde{s} \pm \sqrt{\tilde{s}^2 - \frac{b_{12}^2}{|a_{11}a_{22}|}k_y^2}. \tag{10}$$

Now if we consider a term of the usual form (in $\tilde{s}$), suppose both $\tilde{s}$ and k_y are large with $\tilde{s}$ imaginary, and set:

$$\tilde{s} = ik_y\tau,$$

the damping exponent approximately becomes:

$$\frac{\mu\tau \pm \sqrt{\tau^2+1} - \gamma_1}{\tau + \alpha_1} - \gamma_0. \tag{11}$$

There is, in fact, no way to choose γ_1 and α_1 so that this is positive; near $\tau = -\alpha_1$ the denominator changes sign while the numerator cannot change sign for both signs of the radical term.

Thus we conclude that in the second case we can achieve damping only by adding on a lower order term, γ_0, of the appropriate sign, except in the case $b_{12} = 0$ considered earlier.

5. Geometric Interpretation of the Layer Parameters

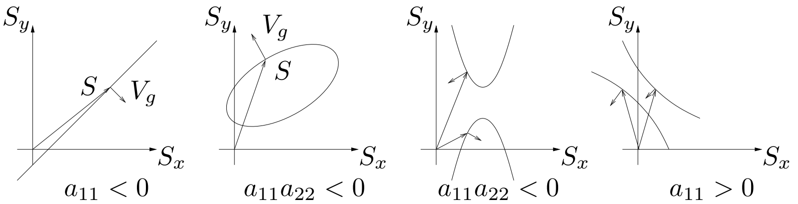
Inspired by the analysis in [2] we now give a geometric interpretation of the layer parameters in terms of the slowness curve. The slowness curve is obtained by considering wave like solutions

$$u = e^{i(k_x x + k_y y - \omega t)},$$

to the system (1). The dispersion relation is then given by $F(\omega, k) \equiv \det(i\omega I + ik_x A + ik_y B) = 0$. Since $F(\omega, k)$ is homogeneous of degree one we have that

$$\begin{aligned} F(1, \frac{k}{\omega}) \equiv 1 + (a_{11}b_{22} + b_{11}a_{22})(\frac{k_y}{\omega})(\frac{k_x}{\omega}) + a_{11}a_{22}(\frac{k_x}{\omega})^2 \\ +(b_{11}b_{22} - b_{12}^2)(\frac{k_y}{\omega})^2 + (a_{22} + a_{11})(\frac{k_x}{\omega}) + (b_{22} + b_{11})(\frac{k_y}{\omega}) = 0. \end{aligned} \tag{12}$$

The slowness curve is defined as the set of points in the plane of slowness vectors $S \equiv (S_x, S_y) = (k_x/\omega, k_y/\omega)$ satisfying (12). In Figure 1 some typical configurations of the slowness curve are plotted. It can be shown (see e.g. [2]) that the group velocity $V_g = (\partial_{k_x}\omega, \partial_{k_y}\omega)$ is orthogonal to the slowness curve. The necessary



$a_{11} < 0$

$a_{11}a_{22} < 0$
$b_{12}^2 > b_{11}b_{22}$

$a_{11}a_{22} < 0$
$b_{12}^2 < b_{11}b_{22}$

$a_{11} > 0$
$a_{22} > 0$

FIGURE 1. Examples of slowness curves for the system (1).

condition for stability for a split-field PML model is that S_x and $\partial_{k_x}\omega$ have the same sign for all S_x on the slowness curve. None of the the slowness curves in Figure 1 satisfy the condition.

The standard construction of split-field PMLs omits the parameters γ_1, γ_0 and α_1. In this section we will take the point of view that their presence is equivalent to a coordinate transformation in the layer which symmetrizes the slowness curve with respect to the coordinate axes. Precisely, the transformed dual variables are given by inverting:

$$\begin{aligned} \omega &= \omega' + a^y k_y', \\ k_x &= k_x' + g^y k_y' - \bar{\gamma}_0 \omega' \\ k_y &= k_y', \end{aligned}$$

which is easily seen to follow from a linear (x,t)-dependent coordinate transformation. (Similar transformations have been used in the context of PMLs previously. For example see [1], [12], and [7].)

5.1. The Transport Equation

The slowness curve for the two dimensional transport equation (3) is the straight line described by $1 + a_{11}S_x + b_{11}S_y = 0$. To satisfy the group velocity criterion we need to make the line parallel to the $S_y^{'}$-axis. With the parameters chosen in Section 3, the slowness curve in the transformed system is given by $1 - a_{11}\bar{\gamma}_0 + a_{11}S_x' = 0$ which is as required. Moreover, by (5) the line lies on the correct side of the S_y'-axis.

5.2. The General Case $b_{12} \neq 0$

Now the slowness curve in the transformed variables satisfies the equation:

$$\left(1 - \frac{(a_{11}+a_{22})^2}{4a_{11}a_{22}}\right) + a_{11}a_{22}S_x^{'2} - b_{12}^2 S_y^{'2} = 0. \tag{13}$$

In the case of waves moving both ways, that is when $a_{11}a_{22} < 0$, the slowness curve is an ellipse whose major and minor axes are aligned with the S_x'- and S_y'-axes, and the group velocity criterion is satisfied. Note that this is true for both the second and third examples in Figure 1; in the third case the hyperbola is transformed into an ellipse. On the other hand, for waves moving one way, which corresponds to the fourth example in Figure 1, the transformed slowness curve is also a hyperbola. Here it is not the group velocity criterion that fails but the fact that the transformation destroys the hyperbolicity of the system with respect to the new time variable.

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