PERFECTLY MATCHED LAYERS FOR HYPERBOLIC SYSTEMS: GENERAL FORMULATION, WELL-POSEDNESS AND STABILITY

DANIEL APPELÖ[†§], THOMAS HAGSTROM[‡¶], AND GUNILLA KREISS[†‖]

Abstract. Since its introduction the Perfectly Matched Layer (PML) has proven to be an accurate and robust method for domain truncation in computational electromagnetics. However, the mathematical analysis of PMLs has been limted to special cases. In particular, the basic question of whether or not a stable PML exists for arbitrary wave propagation problems remains unanswered. In this work we develop general tools for constructing PMLs for first order hyperbolic systems. We present a model with many parameters which is applicable to all hyperbolic systems, and which we prove is well-posed and perfectly matched. We also introduce an automatic method for analyzing the stability of the model and establishing energy inequalities. We illustrate our techniques with applications to Maxwell's equations, the linearized Euler equations, as well as arbitrary 2×2 systems in $(2+1)$ dimensions.

Key words. Perfectly matched layers, stability.

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1. Introduction.

Many important wave propagation problems are posed on unbounded or large domains. Such problems must be solved on a truncated domain if numerical methods are to be used. There exist many techniques for truncating the original domain (see the review papers [14, 15, 23]), but one that has proved both efficient and accurate is the perfectly matched layer (PML) technique. The PML technique surrounds the domain where the solution is desired (the computational domain) by an artificial layer. The layer is constructed so that waves traveling across the interface between the layer and the computational domain are not reflected; that is, the layer is perfectly matched. Moreover, the layer is constructed so that, inside the layer, the solution decays exponentially in the direction normal to the interface. Hence, if the layer is sufficiently wide the solution will be close to zero at the outer boundary and therefore any stable boundary condition can be used there.

Besides the perfect matching and damping properties of the layer it is also desirable that the equations governing the PML be well-posed. This is especially important if a PML derived for a linear problem is to be applied to a non-linear problem or a problem with variable coefficients. If the linearized problem is only weakly well-posed the corresponding non-linear or variable coefficient problem can be ill-posed, see [20]. Well-posedness, by definition, allows the solution to grow exponentially in time and therefore, for a PML to be practically useful, it must also be stable (in time). To summarize, the key properties of a PML are: perfect matching, well-posedness and

[†]Department of Numerical Analysis and Computer Science, Royal Institute of Technology, Stockholm, S-100 44, Sweden.

[‡]Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, NM 87131, USA.

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stability.

PMLs were originally introduced for Maxwell's equations by Bérenger [8]. Well-posedness and stability of the Bérenger PML has been the topic of numerous works. For example Abarbanel and Gottlieb [1] showed that Bérenger's "split-field" PML was only weakly well-posed and that it supported linearly growing modes. Similar results were also obtained via Fourier and energy techniques by Bécache and Joly in [6]. The issue of weak well-posedness led to the development of various well-posed "physical" or "un-split" PMLs for Maxwell's equations; see [2, 13, 24]. These "un-split" PMLs were further improved by the inclusion of the so called complex frequency shift (CFS) which has been used by Bécache et al. [7] to remove late-time linear growth.

For other applications such as the linearized Euler equations [18], the linearized shallow water equations [22], and anisotropic elasticity, [10], there have been reports of exponentially growing solutions. In [3] Abarbanel et al. found that a stable PML could be derived for the linearized Euler equations by transforming the equations into a system whose dispersion relation resembled the dispersion relation of Maxwell's equations. The same transform was later used again to develop a stable PML for the linearized Euler equations [19, 11] and for the linearized shallow water equations [22].

Today there exist stable PML models for many important problems but there are also problems, e.g anisotropic elasticity and linearized MHD, for which stable PMLs have not yet been found. An open issue, then, is whether stable PMLs can be constructed in general. Also, stability and well-posedness for general hyperbolic systems has received less attention than particular cases. One exception is the paper [5] where Bécache et al. give necessary conditions for stability of the split-field PML in terms of the geometrical properties of the dispersion relation. Also, in [4] we construct stable PMLs for arbitrary 2×2 symmetric hyperbolic systems in $2 + 1$ dimensions.

In this work we generalize the formulation of PML models for hyperbolic systems introduced in [16]. To make the model suitable for future applications, we introduce a very general formulation including many free parameters. One of these parameters adds a parabolic term in the tangential directions. By including this parameter we can show that the equations of the PML are well-posed as long as the original hyperbolic system is well posed. In addition we give a proof that the layer is perfectly matched.

We also study the stability of our PML model. The question of stability is not trivial and in general it has to be investigated separately for each new application. To simplify these investigations we introduce a technique, based on criteria for the number of zeros of a polynomial in a half-plane, that can be used to derive necessary and sufficient conditions for stability of any first order constant coefficient Cauchy problem. Moreover, if these conditions are fulfilled there is also a local energy density that decays with time (see [17]). This energy density is automatically generated from the necessary and sufficient conditions. We use the technique to derive stability results for three interesting applications of our general model.

The rest of this paper will be organized as follows. In §2 we present the general PML model for symmetric hyperbolic systems and show that it is perfectly matched and well-posed. In §3 we introduce techniques from [17] used to determine the stability of a first order system with constant coefficients. If the system is stable the technique will yield an energy with a local density that decays with time. In §4 we analyze the stability of a PML model for Maxwell's equation in 2D. The PML is constructed by using the general PML model described in §2. We use the techniques from §3 to establish the stability of the PML and list two associated energies. In §5 we analyze a PML model for the linearized Euler equations and show that it is stable. In §6 we consider the specialization of our general PML formulation to 2×2 symmetric hyperbolic systems in $2+1$ dimensions. In [4] we argued how to choose the layer parameters as functions of the coefficient matrices. Here we prove that these choices will lead to a stable PML. In §7 we conclude and discuss some possible extensions of the presented work.

2. A General Perfectly Matched Layer. We consider the symmetric hyperbolic system in d dimensions:

$$\frac{\partial u}{\partial t} + A_x \frac{\partial u}{\partial x} + \sum_{l=1}^{d-1} A_{y_l} \frac{\partial u}{\partial y_l} + Cu = 0, \tag{1}$$

with initial data, u_0, supported in $-H < x < -h$, $h > 0$. Here $A_x = A_x^T$ and $A_{y_l} = A_{y_l}^T$. For simplicity we assume A_x is invertible; if A_x is singular we only apply the PML to the equations involving x derivatives. We also consider the general PML model:

$$\frac{\partial u}{\partial t} + A_x \left((1+\sigma\eta)\frac{\partial u}{\partial x} + \sigma(\sum_{l=1}^{d-1} \xi_l \frac{\partial u}{\partial y_l} + \mu u) + \sum_j \phi_j \right) + \sum_{l=1}^{d-1} A_{y_l} \frac{\partial u}{\partial y_l} + Cu = 0, \tag{2}$$

$$\frac{\partial \phi_j}{\partial t} + \sigma\phi_j + \alpha_j \phi_j + \sum_{l=1}^{d-1} \beta_{jl} \frac{\partial \phi_j}{\partial y_l} - \sum_{l=1}^{d-1} \varepsilon_{jl} \frac{\partial^2 \phi_j}{\partial y_l^2} = \sigma \left(\gamma_j \frac{\partial u}{\partial x} + \sum_{l=1}^{d-1} \delta_{jl} \frac{\partial u}{\partial y_l} + \nu_j u \right). \tag{3}$$

Here all the additional parameters are real and we also assume:

$$1 + \sigma\eta > 0, \quad \varepsilon_{jl} \geq 0. \tag{4}$$

2.1. Perfect Matching. To investigate the perfect matching of the layer we consider two problems. In the first, whose solution we denote u_1, (1) holds in $R^d \times R$ and in the second, whose solution is denoted u_2, we suppose that (1) holds in $x < 0$ and that (2) and (3) hold in $x > 0$. We also insist that u_2 be continuous. Our goal is to show that the restrictions of each solution to $x < 0$ are identical; that is the layer is perfectly matched.

We begin by performing a Fourier-Laplace transformation in the tangential directions and in time. The duals of $y = [y_1, \ldots, y_{d-1}]$ are denoted by $k = [k_1, \ldots, k_{d-1}]$ and the dual of t by s. This leads to the problems:

$$A_x \frac{\partial \hat{u}_1}{\partial x} + \left(sI + \sum_l ik_l A_{y_l} + C \right) \hat{u}_1 = \hat{u}_0, \quad x \in R, \tag{5}$$

and in the second case, for $x < 0$:

$$A_x \frac{\partial \hat{u}_2^L}{\partial x} + \left(sI + \sum_l ik_l A_{y_l} + C \right) \hat{u}_2^L = \hat{u}_0, \tag{6}$$

and for $x > 0$:

$$\begin{aligned} A_x &\left((1+\sigma\eta) \frac{\partial \hat{u}_2^R}{\partial x} + \sigma(\sum_l ik_l \xi_l + \mu)\hat{u}_2^R + \sum_j \hat{\phi}_j \right) \\ &+ \left(sI + \sum_l ik_l A_{y_l} + C \right) \hat{u}_2^R = 0, \end{aligned} \tag{7}$$

$$\left(s+\sigma+\alpha_j+\sum_l ik_l\beta_{jl}+\sum_l \varepsilon_{jl}k_l^2\right)\hat{\phi}_j=\sigma\left(\gamma_j\frac{\partial \hat{u}_2^R}{\partial x}+(\sum_l ik_l\delta_{jl}+\nu_j)\hat{u}_2^R\right). \quad (8)$$

The solution of (5) follows from the solution of the eigenvalue problem:

$$\lambda A_x w+\left(sI+\sum_l ik_l A_{y_l}+C\right)w=0. \quad (9)$$

We note that for $\Re s>|C|_2$ the eigenvalues, λ, cannot be purely imaginary. In particular if we normalize w to have length one a straightforward computation yields:

$$\Re\lambda=-\frac{\Re s+\Re w^*Cw}{w^*A_xw}, \quad (10)$$

which implies:

$$|\Re\lambda|>(\rho(A_x))^{-1}(\Re s-|C|_2). \quad (11)$$

Thus taking $\Re s$ sufficiently large we may assume that solutions of (9) fall into two sets labeled by the sign of the real parts of the eigenvalues:

$$\Re\lambda_1,\ldots,\Re\lambda_r<0, \quad (12)$$

$$\Re\lambda_{r+1},\ldots,\Re\lambda_n>0. \quad (13)$$

Moreover, the matrix:

$$M(s,k)=-A_x^{-1}\left(sI+\sum_l ik_lA_{y_l}+C\right), \quad (14)$$

can be block diagonalized:

$$QMQ^{-1}=\begin{pmatrix} S^- & 0\\ 0 & S^+\end{pmatrix}, \quad (15)$$

where the eigenvalues (12) are the eigenvalues of S^- and the eigenvalues (13) are the eigenvalues of S^+. Now the bounded solution of (5) is easy to write down:

$$\hat{u}_1=Q^{-1}\begin{pmatrix} \int_{-\infty}^x e^{S^-(x-y)}f^-(y)dy\\ -\int_x^\infty e^{S^+(x-y)}f^+(y)dy\end{pmatrix}, \quad (16)$$

where:

$$QA_x^{-1}\hat{u}_0=\begin{pmatrix} f^-\\ f^+\end{pmatrix}. \quad (17)$$

In particular the support properties of $\hat{u}_0$ and thus $f^\pm$ guarantee the existence of the integrals in (16). We note that at $x=0$:

$$\hat{u}_1=Q^{-1}\begin{pmatrix} \int_{-H}^{-h} e^{-S^-y}f^-(y)dy\\ 0\end{pmatrix}. \quad (18)$$

We now compute $\hat{u}_2$ in each region. We first note that equation (8) can be solved directly:

$$\hat{\phi}_j = \frac{\sigma\left(\gamma_j \frac{\partial \hat{u}_2^R}{\partial x} + (\sum_l i k_l \delta_{jl} + \nu_j)\hat{u}_2^R\right)}{s + \sigma + \alpha_j + \sum_l i k_l \beta_{jl} + \sum_l \varepsilon_{jl} k_l^2}. \tag{19}$$

Now for $x > 0$ we transform the solution using the same transformation Q which block diagonalizes the problem for $x < 0$. Setting $v = Q\hat{u}_2^R$ we find:

$$v_x = \frac{1}{r(s,k) + \sigma p(s,k)} \begin{pmatrix} r(s,k)S^- - \sigma q(s,k) I & 0 \\ 0 & r(s,k)S^+ - \sigma q(s,k) I \end{pmatrix} v, \tag{20}$$

where the polynomials r, p and q are defined up to a constant multiple by:

$$\eta + \sum_j \frac{\gamma_j}{s + \sigma + \alpha_j + \sum_l i k_l \beta_{jl} + \sum_l \varepsilon_{jl} k_l^2} = \frac{p(s,k)}{r(s,k)}, \tag{21}$$

$$\sum_l i k_l \xi_l + \mu + \sum_j \frac{\sum_l i k_l \delta_{jl} + \nu_j}{s + \sigma + \alpha_j + \sum_l i k_l \beta_{jl} + \sum_l k_l^2 \varepsilon_{jl}} = \frac{q(s,k)}{r(s,k)}. \tag{22}$$

We will argue that for $\Re s$ sufficiently large these blocks have eigenvalues with negative and positive real parts respectively. In particular we note that:

$$\lim_{|s| \to \infty} \frac{p}{r} = \eta, \quad \lim_{|s| \to \infty} \frac{q}{r} = \sum_l i k_l \xi_l + \mu. \tag{23}$$

Thus for large s the eigenvalues are approximately:

$$\frac{\lambda_j - \sigma(\sum_l i k_l \xi_l + \mu)}{1 + \sigma\eta}. \tag{24}$$

Now by (11) and (4) we conclude that the signs of their real parts are the same as the signs of $\Re\lambda_j$ if we choose $\Re s$ sufficiently large, which was what we wished to prove.

From this argument we conclude that the transform of the causal solution in $x > 0$ takes the form:

$$\hat{u}_2^R = Q^{-1} \begin{pmatrix} e^{(r+\sigma p)^{-1}(rS^- - \sigma q I)x} v^- \\ 0 \end{pmatrix}. \tag{25}$$

We see that this can be "perfectly matched" to the restriction of $\hat{u}_1$ to $x < 0$ by setting:

$$v^- = \int_{-H}^{-h} e^{-S^- y} f^-(y) dy. \tag{26}$$

Thus we have proven that u_1 and u_2 restricted to $x < 0$ are identical.

We note that we can interpret the layer as an (s,k)-dependent change of variables:

$$\hat{u}(x) \to e^{-a\tilde{x}} \hat{u}(\tilde{x}), \tag{27}$$

where

$$\tilde{x} = \frac{r}{r + \sigma p} x, \quad a = \sigma \frac{q}{r}. \tag{28}$$

With this interpretation the new layer can be viewed as a generalization of the Bérenger layer from the viewpoint of complex coordinate stretching as introduced by Chew and Weedon [9].

2.2. Well-posedness of the Layer Equations. For the applications considered in this paper it will be sufficient to include only one set of auxiliary variables, leading to the PML model

$$\begin{aligned}
&\frac{\partial u}{\partial t} + A_x\left((1+\sigma\eta)\frac{\partial u}{\partial x} + \sigma(\sum \xi_l \frac{\partial u}{\partial y_l} + \mu u) + \phi\right) + \sum A_{y_l}\frac{\partial u}{\partial y_l} + Cu = 0,\\
&\frac{\partial \phi}{\partial t} + \sigma\phi + \alpha\phi + \sum \beta_l \frac{\partial \phi}{\partial y_l} - \sum \varepsilon_l \frac{\partial^2 \phi}{\partial y_l^2} = \sigma\left(\gamma\frac{\partial u}{\partial x} + \sum \delta_l \frac{\partial u}{\partial y_l} + \nu u\right).
\end{aligned} \tag{29}$$

Even with just one set of auxiliary variables, there are many free parameters that must be chosen. Our experience is that the parameters $\mu, \xi, \beta_l, \delta_l, \gamma$ can be determined from the coefficients of the matrices A_x and A_y. The parameter η is introduced in the model to increase the damping of evanescent modes. The parameter α, which is usually referred to as the complex frequency shift (CFS), typically enhances stability properties at late time.

To our knowledge the parabolic terms $\varepsilon_l \phi_{y_l y_l}$ (hereafter called parabolic CFS) have not been included in PML models before. We have choosen to include them to guarantee the well-posedness of the model (29). To see this we freeze the coefficients and perform a Fourier transform in space (k_x is the dual of x). Excluding the zero order terms in the symbol of the equations (29) we obtain

$$P_1(ik) = -\begin{bmatrix} (ik_x(1+\sigma\eta) + \sum ik_l\xi_l\sigma)A_x + \sum ik_l A_{y_l} & 0 \\ -(ik_x\sigma\gamma + \sum ik_l\delta_l\sigma)I & \sum \varepsilon_l k_l^2 I + \sum ik_l\beta_l I \end{bmatrix}. \tag{30}$$

Denote the upper diagonal block in P_1, (30), by P_{11}. By the hyperbolicity of the original problem P_{11} is diagonalizable with imaginary eigenvalues.

Without the parabolic complex frequency shift the lower diagonal block also has purely imaginary eigenvalues, but the system may be only weakly hyperbolic. This is the case if, for some set of $k_1, \ldots, k_{d-1}$,

$$\sum_l ik_l\beta_l, \tag{31}$$

coincides with one of P_1's eigenvalues while

$$k_x\sigma\gamma + \sum k_l\delta_l\sigma \neq 0. \tag{32}$$

Then it is not possible to diagonalize P_1, and the problem is not well-posed. Otherwise the system is strongly hyperbolic and thus well-posed.

If all $\varepsilon_l \neq 0$ the lower diagonal block always has eigenvalues that are distinct from the eigenvalues of P_{11}, as shown by the following argument. When at least one k_l is non-zero, the eigenvalue of the lower block has negative real part. For $k_1 = \ldots = k_{d-1} = 0$, P_{11} is nonsingular since A_x is nonsingular, while the lower diagonal block is zero. It follows that P_1 is always diagonalizable and the system is well-posed. This proves

LEMMA 1. *If $\varepsilon_l > 0$, $l = 1, \ldots, d-1$, and the original system (1) is well-posed, then the PML (29) is also well-posed.*

We conclude this section by noting that the PML for many problems is well-posed without the parabolic CFS. Additionally, if the parabolic CFS is used, ε_l should be chosen relative to the grid size such that it does not impose restrictions on the time-stepping.

3. Construction of Energy Estimates for Constant Coefficient Cauchy Problems via Annihilating Polynomials. As we have seen in the previous section, the construction of a layer which is well-posed and perfectly matched is rather straightforward. However, it is not so straightforward to choose the free parameters η, ξ, μ, α_j, β_{jl}, δ_{jl}, ε_{jl} and ν_j, for a given hyperbolic system, such that the solution does not grow with time. Related to this question is the stability of the constant coefficient Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = P(\partial/\partial x)u(x,t), \quad u(x,0) = u_0(x), \quad x \in \mathbf{R}^d, \quad 0 \geq t \geq T. \tag{33}$$

If we perform a Fourier transform in space (33) reduces to a system of ordinary differential equations

$$\frac{\partial \hat{u}(k,t)}{\partial t} = P(ik)\hat{u}(k,t), \qquad k \in \mathbf{R}^s, \quad 0 \geq t \geq T, \tag{34}$$

$$\hat{u}(k,0) = \hat{u}_0(k). \tag{35}$$

We will distinguish between the following two types of stability.

Definition 2 (Stability). *We say that the Cauchy problem (33) is*

(i) ***strongly stable****: if all solutions satisfy an estimate* $\|u(\cdot,t)\|_{L^2} \leq K\|u_0(\cdot)\|_{L^2}$*;*

(ii) ***weakly stable****: if the solutions satisfy an estimate* $\|u(\cdot,t)\|_{L^2} \leq K(1+t)^p\|u_0(\cdot)\|_{H^s}$*, where* $s > 0$.

Note that if (33) is well-posed we can replace H^s by L^2 in (ii). In the remainder of this paper we will drop the subscript of the L^2-norm, i.e. $\|\cdot\| \equiv \|\cdot\|_{L^2}$.

A necessary and sufficient condition for weak stability is that all eigenvalues λ_j of the symbol $P(ik)$ satisfy

$$\Re\{\lambda_j(P(ik))\} \leq 0. \tag{36}$$

Condition (36) can be checked by various methods that determine the number of zeros of polynomials in a half-plane. Below, we will first present a method that automatically generates a finite number of algebraic inequalities that can be used to check (36). Then we will show that if (36) holds, the method can also be used to construct a local energy density that decays with time.

We begin by recalling some definitions from matrix theory (see e.g. [12]).

Definition 3 (Annihilating Polynomial). *We say that a scalar polynomial* $f(\lambda)$ *is an annihilating polynomial of the square matrix* A *if*

$$f(A) = 0.$$

Two important annihilating polynomials are the characteristic polynomial and the minimal polynomial.

Definition 4 (Characteristic Polynomial). *The scalar polynomial* $f(\lambda)$ *defined as*

$$f(\lambda) \equiv \det((\lambda I - A),$$

is called the characteristic polynomial of the matrix A.

Definition 5 (Minimal Polynomial). *By* $m_A(\lambda)$ *we will denote the uniquely defined annihilating polynomial of lowest degree and with lead coefficient 1. The polynomial* $m_A(\lambda)$ *is called the minimal polynomial of* A.

Now, let $m_P(\lambda)$ be the minimal polynomial of the symbol $P(ik)$. Suppose its degree is n. To determine the number of roots with positive and negative real part of $m_P(\lambda) = 0$ for fixed k we can use the following lemma, which is a special case of Corollary (38,1b) in [21].

LEMMA 6. *Consider any polynomial $q(\lambda)$ of degree n. Let D be a real number and define the polynomials Q_0 and Q_1 with real coefficients by*

$$q(iD) \equiv i^n \left[Q_0(D) + iQ_1(D)\right]. \tag{37}$$

Then there is a continued fraction

$$\frac{Q_1(D)}{Q_0(D)} = \cfrac{1}{c_1D + d_1 - \cfrac{1}{c_2D + d_2 - \cfrac{1}{c_3D + d_3 - \cdots - \cfrac{1}{c_{n_r}D + d_{n_r},}}}} \tag{38}$$

with $c_j \neq 0$ and $n_r \leq n$. The number of roots with positive (negative) real part equals the number of positive (negative) c_j. There are $n - n_r$ roots on the imaginary axis.

When we apply Lemma 6 to $m_p(\lambda)$, the number of nonzero coefficients c_j may depend on k. A change in sign corresponds to a root crossing the imaginary axis. We have

COROLLARY 7. *A necessary and sufficient condition for weak stability is that all c_j defined in (38) are negative, i.e.*

$$c_j(k) < 0, \quad j = 1, 2, \ldots, n_r(k). \tag{39}$$

Remark. Strong stability follows if all eigenvalues (i.e. the roots of $m_P(\lambda) = 0$) have strictly negative real part for all k. However, in many cases there are certain k for which some roots have zero real part. If the corresponding eigenvectors span their respective invariant subspace then the problem is still strongly stable. This condition must be checked in each case. We note that if (33) is well-posed then, for sufficiently large $|k| \geq K$, $P(ik)$ can always be diagonalized. Thus, we only need to check the eigenvectors for roots that have zero real part at bounded $|k|$.

PROPOSITION 8. *Let $\hat{u}$ be the solution of the Fourier transformed system (34). Then any component $\hat{u}_i$ satisfies the equation*

$$g(\frac{\partial}{\partial t})\hat{u}_i = 0,$$

where $g(\lambda)$ is any annihilating polynomial of the symbol, $P(ik)$. In particular we have for the minimal polynomial of $P(ik)$

$$m_P(\frac{\partial}{\partial t})\hat{u}_i(k,t) = 0. \tag{40}$$

Proof. By definition we have $g(P(ik)) = 0$. Multiplying by the solution vector from the right yields $g(P(ik))\hat{u} = 0$. By an easy induction argument we have for any integer q, $(P(ik))^q\hat{u} = \frac{\partial^q \hat{u}}{\partial t^q}$. The proposition follows. ∎

By the following theorem we can construct decaying energies for the problem (33).

THEOREM 9. *Let $\hat{u}_i$ satisfy*

$$q(\frac{\partial}{\partial t})\hat{u}_i = 0. \tag{41}$$

If (39) holds for Q_0 and Q_1 defined as in lemma 6, there exists an energy

$$\mathcal{E}(t;k) \equiv \frac{1}{2}\sum_{j=1}^{n_r} |c_j||\hat{z}^{(j)}(k,t)|^2, \tag{42}$$

satisfying

$$\frac{\partial}{\partial t}\mathcal{E}(t;k) = -|\hat{z}^{(1)}(k,t)|^2. \tag{43}$$

The functions $\hat{z}^{(j)}, j = 1,\dots,n$, are related to $\hat{u}_i(k,t)$ via the equations

$$\begin{gathered}\prod_{j=1}^{n-n_r}(\frac{\partial}{\partial t} + ib_j(k))\hat{u}_i(k,t) = -i\hat{z}^{(1)}(k,t), \quad \Im b_j(k) = 0,\\ \frac{\partial}{\partial t}\begin{bmatrix} |\, c_1|\hat{z}^{(1)} \\ \vdots \\ \vdots \\ |\, c_{n_r}|\hat{z}^{(n_r)} \end{bmatrix} = \begin{bmatrix} id_1 - 1 & -i & \cdots & 0 \\ -i & id_2 & \cdots & 0 \\ 0 & \vdots & \ddots & -i \\ 0 & \vdots & -i & id_{n_r} \end{bmatrix} \begin{bmatrix} \hat{z}^{(1)} \\ \vdots \\ \vdots \\ \hat{z}^{(n_r)} \end{bmatrix}.\end{gathered} \tag{44}$$

For the proof of Theorem 9 we refer to [17]. Note that system (44) can be used to eliminate all $\hat{z}^{(j)}$ so that the energy (42) is expressed in $\hat{u}_i$ alone.

3.1. Localization of $\mathcal{E}(t;k)$. Since the coefficients c_j are rational functions in k localization can by accomplished by multiplication with a suitable polynomial in k. In particular let $\gamma(k)$ be a polynomial such that $\tilde{c}_j(k) = -\gamma(k)^2 c_j(k)$ is also a polynomial in k. Then, since by assumption $\tilde{c}_j(k) > 0$ for all k, it can be decomposed as

$$\tilde{c}_j(k) = \sum_l q_l^2(k), \tag{45}$$

where $q_l(k)$ are real polynomials in k. By multiplying (43) with $\gamma(k)^2$ we get

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\sum_{j=1}^{n_r}|\gamma(k)^2 c_j(k)|\,|\hat{z}^{(j)}(t,k)|^2 &= \frac{d}{dt}\frac{1}{2}\sum_{j=1}^{n_r}\sum_l \left|q_l(k)\,\hat{z}^{(j)}(t,k)\right|^2 \\ &= -\left|\gamma(k)\,\hat{z}^{(1)}(t,k)\right|^2.\end{aligned} \tag{46}$$

Integrating over k yields

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\sum_{j=1}^{n_r}\sum_l \int_{\mathbb{R}^s}\left|q_l(k)\,\gamma(k)\,\hat{z}^{(j)}(t,k)\right|^2\,dk \\ &\quad = -\int_{\mathbb{R}^s}\left|\gamma(k)\,\hat{z}^{(1)}(t,k)\right|^2\,dk,\end{aligned}$$

and by applying Parseval's formula $\int |\hat{f}(k)|^2 dk = \|f(x)\|^2$ we have the following:

COROLLARY 10. *There exists a polynomial $\gamma(k)$ such that the inverse transform of (46) is*

$$\frac{d}{dt}E(t) = -\left\|\mathcal{F}^{-1}\left\{\gamma(k)\,\hat{z}^{(1)}(t,k)\right\}\right\|^2. \tag{47}$$

Here

$$E(t) = \frac{1}{2}\sum_{j=1}^{n}\sum_{l}\left\|\mathcal{F}^{-1}\left\{q_l(k)\,\hat{z}^{(j)}(t,k)\right\}\right\|^2, \tag{48}$$

contains only local quantities.

Note that Theorem 9 can be used with any annihilating polynomial of $P(ik)$. If the minimal polynomial is available it is advantageous to use it since it has lower degree and thus will produce an energy with lower order derivatives. Its lower degree also simplifies the computation of the continued fraction.

4. PML for Maxwell's Equations. The first problem we consider is the scaled TM_z problem in a lossless medium. Then Maxwell's equations can be written

$$\frac{\partial u}{\partial t} + A_x\frac{\partial u}{\partial x} + A_y\frac{\partial u}{\partial y} = 0,$$
$$A_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad A_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

where $u = [H_x, H_y, E_z]^T$. We consider a layer in the x direction. Here A_x is singular and there are only two modes that propagate in the x direction. Hence, we only add auxiliary variables to the x-propagating H_y and E_z fields. The layer we will consider is defined by the equations

$$\begin{aligned}
\frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial y} &= 0,\\
\frac{\partial H_y}{\partial t} - (1+\eta\sigma)\frac{\partial E_z}{\partial x} &= \sigma\phi_2,\\
\frac{\partial E_z}{\partial t} - (1+\eta\sigma)\frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y} &= \sigma\phi_1,\\
\frac{\partial \phi_1}{\partial t} + \frac{\partial E_z}{\partial x} &= -(\sigma+\alpha)\phi_1 + \varepsilon\frac{\partial^2\phi_1}{\partial y^2},\\
\frac{\partial \phi_2}{\partial t} + \frac{\partial H_y}{\partial x} &= -(\sigma+\alpha)\phi_2 + \varepsilon\frac{\partial^2\phi_2}{\partial y^2}.
\end{aligned} \tag{49}$$

Here we have included the parameter η which will improve the damping of evanescent modes. Note that if $\varepsilon = 0$ the above equations are only weakly hyperbolic and thus only weakly well-posed. To ensure strong well-posedness we take $\varepsilon > 0$.

4.1. Stability for Constant σ. When σ is constant we can take the Fourier transform in x and y. For simplicity let $\eta = 0$. The symbol of (49) then becomes

$$P(ik) = -\begin{bmatrix} ik_xA_x + ik_yA_y & \sigma D \\ ik_xE & (\alpha+\sigma+\varepsilon k_y^2)I \end{bmatrix}, \tag{50}$$

where

$$D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T, \quad E = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The minimal polynomial of (50) coincides with the characteristic polynomial and can be written as a product of the two polynomials $m_P(\lambda) = m_1(\lambda)m_4(\lambda)$

$$\begin{aligned} m_1(\lambda) &= \lambda, \\ m_4(\lambda) &= \big(\lambda^4 + 2(\tau+\sigma)\lambda^3 + (k_x^2 + k_y^2 + (\tau+\sigma)^2)\lambda^2 \\ &\quad + 2(\tau k_x^2 + (\tau+\sigma)k_y^2)\lambda + k_x^2\tau^2 + (\tau+\sigma)^2 k_y^2\big), \end{aligned}$$

where we have introduced $\tau = \alpha + \varepsilon k_y^2$.

To determine the sign of the eigenvalues we apply Lemma 6 to $m_4(\lambda)$. The coefficients in the continued fraction (38) are

$$c_1 = -\frac{1}{2(\tau+\sigma)}, \tag{51}$$

$$c_2 = -\frac{2\,(\tau+\sigma)^2}{(\tau+\sigma)^3 + \sigma k_x^2}, \tag{52}$$

$$c_3 = -\frac{\big((\tau+\sigma)^3 + \sigma k_x^2\big)^2}{2\sigma k_x^2\,\big((\tau+\sigma)(\tau^2+\sigma\tau+k_y^2) + \tau k_x^2\big)\,(\tau+\sigma)}, \tag{53}$$

$$c_4 = -\frac{2\sigma k_x^2\,\big((\tau+\sigma)(\tau^2+\tau\sigma+k_y^2) + \tau k_x^2\big)}{\big((\tau+\sigma)^2 k_y^2 + k_x^2\tau^2\big)\,\big((\tau+\sigma)^3 + \sigma k_x^2\big)}, \tag{54}$$

$$d_1 = d_2 = d_3 = d_4 = 0. \tag{55}$$

Clearly all c_j are negative and defined except for the cases $k_x = k_y = 0$ and $k_x = 0, k_y \neq 0$. When $k_x = 0$ the minimal equation reduces to $\lambda(\lambda^2 + k_y^2)(\lambda + \sigma + \tau)^2 = 0$ with solutions $\lambda = 0, \pm i k_y, -(\sigma+\tau), -(\sigma+\tau)$. The eigenvalues with zero real part are distinct as long as $k_y \neq 0$. For the case $k_y = 0$ there could potentially be algebraic growth. However, it is easily checked that there are three independent eigenvectors when $k_x = k_y = 0$. Thus (49) is strongly stable when $\eta = 0$. When $\eta \neq 0$ the coefficients are somewhat more complicated, but strong stability follows similarly. This concludes the proof of the following lemma.

LEMMA 11. *For constant $\sigma > 0$, $\alpha > 0$, $\varepsilon > 0, \eta\sigma + 1 > 0$, the system (49) is strongly stable.*

4.2. Energy Estimates. We now consider decaying energies of the system (49). We start by noticing that $m_4(P(ik))$ annihilates $\hat{E}_z$ and $\hat{\phi}_2$ while $m_P(\partial/\partial t)$ annihilates $\hat{H}_x$, $\hat{H}_y$ and $\hat{\phi}_1$. Thus we have that

$$m_4(\partial/\partial t)\hat{v} = 0, \quad \text{for} \quad \hat{v} = \hat{E}_z, \hat{\phi}_2, \frac{\partial \hat{H}_x}{\partial t}, \frac{\partial \hat{H}_y}{\partial t}, \frac{\partial \hat{\phi}_1}{\partial t}.$$

It follows from Theorem 9 and (51)-(55) that the energy

$$\mathcal{E}(t;k) \equiv \frac{1}{2}\sum_{j=1}^{4} |c_j||\hat{z}^{(j)}(k,t)|^2, \tag{56}$$

decays with time. To express (56) in $\hat{v}$ we use (44). This yields

$$
\begin{aligned}
|z^{(1)}| &= |\hat{v}|, \quad |z^{(2)}| = \left| \left(|c_1| \frac{\partial}{\partial t} + 1 \right) \hat{v} \right|, \\
|z^{(3)}| &= \left| \left(|c_2| \frac{\partial}{\partial t} \left(|c_1| \frac{\partial}{\partial t} + 1 \right) + 1 \right) \hat{v} \right|, \\
|z^{(4)}| &= \left| \left(|c_3| \frac{\partial}{\partial t} \left(|c_2| \frac{\partial}{\partial t} \left(|c_1| \frac{\partial}{\partial t} + 1 \right) + 1 \right) + \left(|c_1| \frac{\partial}{\partial t} + 1 \right) \right) \hat{v} \right|.
\end{aligned} \tag{57}
$$

Since $\mathcal{E}$ is a function of c_3 and c_4, whose denominators vanish for certain k_x and k_y, it is not bounded. To formulate energies in physical space, we first remove the singularities of $\mathcal{E}$ by multiplying (56) by a suitable polynomial in k_x and k_y. Here we will consider two different polynomials, the first producing a semi-local energy and the second a fully local energy.

4.2.1. A Semi-Local Energy. We would like the order of the spatial derivatives of v appearing in the energy in physical space to be as low as possible. At the same time, the energy must be bounded for all k_x and k_y so that we can use Parseval. The energy

$$\mathcal{E}_{SL}(t;k) = 2(\tau+\sigma)k_x^2 \left((\tau+\sigma)^2 k_y^2 + k_x^2 \tau^2 \right) \mathcal{E}(t;k), \tag{58}$$

satisfies these requirements. We can split $\mathcal{E}_{SL}$ into a local and a non-local part

$$\mathcal{E}_{SL}(t;k) = \mathcal{E}_L + \mathcal{E}_{NL}. \tag{59}$$

The local and non-local energies are

$$\mathcal{E}_L = k_x^2 \left((\tau+\sigma)^2 k_y^2 + k_x^2 \tau^2 \right) |\hat{v}|^2, \tag{60}$$

$$\mathcal{E}_{NL} = 2(\tau+\sigma)k_x^2 \left((\tau+\sigma)^2 k_y^2 + k_x^2 \tau^2 \right) \sum_{j=2}^{4} |c_j| |\hat{z}^{(j)}(k,t)|^2. \tag{61}$$

Now by using Parseval we get

$$\frac{d}{dt} \left(E_L(t;v) + E_{NL}(t) \right) = -2(\sigma+\alpha) E_L(t;v) - 2\varepsilon E_L(t;\partial_y v), \tag{62}$$

where

$$
\begin{aligned}
E_L(t;v) = (\alpha+\sigma)^2 \|\partial_x \partial_y v(\cdot,t)\|^2 + \alpha^2 \|\partial_x^2 v(\cdot,t)\|^2 + 2\varepsilon(\alpha+\sigma) \|\partial_x \partial_y^2 v(\cdot,t)\|^2 \\
+ \varepsilon^2 \|\partial_x \partial_y^3 v(\cdot,t)\|^2 + 2\varepsilon\alpha \|\partial_x^2 \partial_y v(\cdot,t)\|^2 + \varepsilon^2 \|\partial_x^3 \partial_y v(\cdot,t)\|^2.
\end{aligned} \tag{63}
$$

We do not state $E_{NL}(t)$ explicitly, since for our purpose it is sufficient to know that it is bounded and non-negative. However, we note that $E_{NL}(t)$ is non-local in space.

By rewriting (62)

$$\frac{d}{dt} \left(e^{2(\sigma+\alpha)t} E_L(t;v) + E_{NL}(t) \right) = -2\varepsilon E_L(t;\partial_y v), \tag{64}$$

we see that

$$E_L(t;v) \le C e^{-2(\sigma+\alpha)t}, \quad C = E_L(0\,;v), \tag{65}$$

which proves the following

LEMMA 12. *Let v be any of the fields*

$$E_z, \phi_2, \frac{\partial H_x}{\partial t}, \frac{\partial H_y}{\partial t}, \frac{\partial \phi_1}{\partial t}.$$

If $\sigma > 0$, $\alpha > 0$, $\varepsilon > 0$ and constant then v satisfies the estimate

$$(\alpha+\sigma)^2\|\partial_x\partial_y v(\cdot,t)\|^2 + \alpha^2\|\partial_x^2 v(\cdot,t)\|^2 \leq e^{-2(\sigma+\alpha)t}C. \tag{66}$$

4.2.2. A Local Energy. To obtain a fully local energy we need to clear the denominators of (56) and (57). Again, this is done by multiplying $\mathcal{E}$ by a suitable factor. For this case we define the fully local energy by

$$\begin{aligned}\mathcal{E}_{FL} \equiv\ & (\tau+\sigma)^2((\tau+\sigma)^3+\sigma k_x^2) \\ & \times \left((\tau+\sigma)^2 k_y^2 + k_x^2\tau^2\right) k_x^2 \left((\tau+\sigma)(\tau^2+\sigma\tau+k_y^2)+\tau k_x^2\right)\mathcal{E}.\end{aligned}$$

$\mathcal{E}_{FL}(t;k)$ can be split into

$$\mathcal{E}_{FL}(t;k) = \mathcal{E}^I + \mathcal{E}^{II} + \mathcal{E}^{III} + \mathcal{E}^{IV},$$

where

$$\begin{aligned}\mathcal{E}^I =& \frac{1}{2}(\tau+\sigma)((\tau+\sigma)^3+\sigma k_x^2)\left((\tau+\sigma)^2k_y^2+k_x^2\tau^2\right) \\ &\times k_x^2\left((\tau+\sigma)(\tau^2+\sigma\tau+k_y^2)+\tau k_x^2\right)|\hat{v}|^2,\end{aligned}$$

$$\begin{aligned}\mathcal{E}^{II} =& 2(\tau+\sigma)^2k_x^2\left((\tau+\sigma)^2k_y^2+k_x^2\tau^2\right) \\ &\times\left((\tau+\sigma)(\tau^2+\sigma\tau+k_y^2)+\tau k_x^2\right)\Big|\underbrace{(\frac{1}{2}\partial_t+\tau+\sigma)\hat{v}}_{\hat{\chi}_1}\Big|^2,\end{aligned}$$

$$\begin{aligned}\mathcal{E}^{III} =& \frac{1}{2\sigma}(\tau+\sigma)((\tau+\sigma)^3+\sigma k_x^2)\left((\tau+\sigma)^2k_y^2+k_x^2\tau^2\right) \\ &\times\Big|\underbrace{\left((\sigma+\tau)\partial_t^2+2(\sigma+\tau)^2\partial_t+(\tau+\sigma)^3+\sigma k_x^2\right)\hat{v}}_{\hat{\chi}_2}\Big|^2,\end{aligned}$$

$$\begin{aligned}\mathcal{E}^{IV} =& \Big|\Big(((\tau+\sigma)^3+\sigma k_x^2)\partial_t(\frac{(\tau+\sigma)}{2\sigma}\partial_t^2+\frac{(\tau+\sigma)^2}{\sigma}\partial_t+\frac{1}{2\sigma}) \\ &+k_x^2\left((\tau+\sigma)(\tau^2+\sigma\tau+k_y^2)+\tau k_x^2\right)(\frac{1}{2}\partial_t+\sigma+\tau)\Big)\hat{v}\Big|^2.\end{aligned}$$

To localize the energies we need to write them in the form (45). This is a straightforward operation, but the resulting expressions become lengthy (they contain many combinations of higher derivatives) and are therefore presented in appendix A.

Let E^n be the physical space version of the energy $\mathcal{E}^n$. Then we have that

$$\frac{d}{dt}\left(E^I(t)+E^{II}(t)+E^{III}(t)+E^{IV}(t)\right) = -E^I(t), \tag{67}$$

which means that

$$\left(e^t E^I(t) + E^{II}(t) + E^{III}(t) + E^{IV}(t)\right) \le \left(E^I(0) + E^{II}(0) + E^{III}(0) + E^{IV}(0)\right).$$

Thus E^I decays exponentially while E^{II}, E^{III} and E^{IV} at least remain bounded.

It may be possible to derive sharper results from this fully local energy by using the system (49). We note that the energy estimates obtained by Bécache et al in [7] are stated in terms of the fields rather than the derivatives of the fields so that strong stability is a straightforward consequence of the energy inequality. We emphasize that our results also imply strong stability even though the energy is stated in terms of derivatives of the fields.

5. The Linearized Euler Equations. The next problem we consider are the Euler equations in two dimensions linearized around a subsonic skew flow

$$\frac{\partial u}{\partial t} + A_x \frac{\partial u}{\partial x} + A_y \frac{\partial u}{\partial y} = 0,$$

where

$$u = \begin{bmatrix} \rho \\ v_x \\ v_y \\ p \end{bmatrix}, \quad A_x = \begin{bmatrix} M_x & 1 & 0 & 0 \\ 0 & M_x & 0 & 1 \\ 0 & 0 & M_x & 0 \\ 0 & 1 & 0 & M_x \end{bmatrix}, \quad A_y = \begin{bmatrix} M_y & 0 & 1 & 0 \\ 0 & M_y & 0 & 0 \\ 0 & 0 & M_y & 1 \\ 0 & 0 & 1 & M_y \end{bmatrix}.$$

Here, ρ is the density, v_x and v_y are the velocities in the x and y direction respectively, p is the pressure and M_x and M_y are the Mach numbers in the x and y direction. We have that $0 < M_x < 1, 0 < M_y < 1$ since the flow is assumed to be subsonic.

From [16] we conclude that a suitable layer in the x-direction should be of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + A_x \left(\frac{\partial u}{\partial x} + \mu\sigma u + \sigma\phi\right) + A_y \frac{\partial u}{\partial y} = 0, \\ &\frac{\partial \phi}{\partial t} + \frac{\partial u}{\partial x} + M_y \frac{\partial \phi}{\partial y} + (\sigma + \alpha)(\mu u + \phi) = 0. \end{aligned} \tag{68}$$

The symbol $P(ik)$ of (68) is

$$P(ik) = -\begin{bmatrix} (ik_x + \mu\sigma)A_x + ik_y A_y & \sigma A_x \\ ik_x I + (\sigma + \alpha)\mu I & (ik_y M_y + \sigma + \alpha) I \end{bmatrix}. \tag{69}$$

Note that here we do not need to include the parabolic CFS. To establish well-posedness, we simply freeze the coefficients and consider the principal part of $P(ik)$

$$P_1(ik) = -\begin{bmatrix} ik_x A_x + ik_y A_y & 0 \\ ik_x I & ik_y M_y \end{bmatrix}. \tag{70}$$

The eigenvalues of the upper diagonal block are easy to compute. They coincide with $ik_y M_y$ only when $k_x = 0$. Thus P_1 is diagonalizable and well-posedness follows.

5.1. Stability for Constant σ. In [16] it was shown that the choice

$$\mu = \frac{M_x}{1 - M_x^2}, \tag{71}$$

is necessary for the solution in a layer closely related to (68) to decay in space. Similar conclusions, from another point of view, were reached by Hu in [19]. The results for decay in space from [16] apply directly to (68). Here we will show that (71) is also necessary and sufficient for stability (in time) for (68).

First we show that (71) is sufficient. We note that the real part of the eigenvalues of $P(ik)$ (with μ given by (71)) coincide with the real part of the eigenvalues of the matrix $\tilde{P}(ik) \equiv P(ik) - ik_y M_y I$. Since $\tilde{P}(ik)$ has a sparser structure, it is easier to check that its eigenvalues have non-positive real part.

The minimal polynomial of $\tilde{P}(ik)$ can be factored $m_{\tilde{P}}(\lambda) = m_1(\lambda)m_2(\lambda)$, where

$$m_1(\lambda) = \lambda^2 + \left(ik_x M_x + \frac{\sigma}{\zeta} + \alpha\right)\lambda + ik_x\alpha M_x, \tag{72}$$

and

$$\begin{aligned} m_2(\lambda) =&\lambda^4 + 2\left(ik_x M_x + \frac{\sigma}{\zeta} + \alpha\right)\lambda^3 \\ &+ \left(4\alpha i k_x M_x + \zeta k_x^2 + k_y^2 + \frac{(\sigma+\alpha)^2 - M_x^2\alpha^2}{\zeta}\right)\lambda^2 \\ &+2\left(ik_x\alpha M_x + \zeta\alpha k_x^2 + (\alpha+\sigma)k_y^2\right)\lambda + \zeta\alpha^2 k_x^2 + (\alpha+\sigma)^2 k_y^2. \end{aligned} \tag{73}$$

Here we have introduced $\zeta = 1 - M_x^2$. The continued fraction coefficients for (72) are

$$c_1 = -\frac{\zeta}{2(\sigma+\alpha\zeta)}, \tag{74}$$

$$c_2 = -\frac{2(\sigma+\alpha\zeta)^3}{\alpha\sigma M_x^2\zeta^2 k_x^2}. \tag{75}$$

For (73) the coefficients are

$$c_1 = -\frac{\zeta}{2(\sigma+\alpha\zeta)}, \tag{76}$$

$$c_2 = -\frac{2(\sigma+\alpha\zeta)^3}{c_{2a}}, \tag{77}$$

$$c_3 = -\frac{c_{2a}^3}{2\sigma\zeta(\sigma+\alpha\zeta)^4 c_{3a}}, \tag{78}$$

$$c_4 = -\frac{2(\alpha\zeta+\sigma)^4 c_{3a}^3\sigma}{c_{2a}^4(k_y^2M_x^2 - \zeta k_x^2)^2(\alpha^2\zeta k_x^2 + (\alpha+\sigma)^2 k_y^2)c_{4a}}, \tag{79}$$

where c_{2a}, c_{3a}, c_{4a} are positive for all k_x and k_y and can be found in Appendix B. We see that all the coefficients are negative and defined for all k_x and k_y except the cases (a) $k_x = k_y = 0$, (b) $k_x = 0$, $k_y \neq 0$ and (c) $(1-M_x^2)k_x^2 = M_x^2 k_y^2$. We will consider these cases separately.

First we consider the case (a) for which we easily can compute the eigenvalues of $\tilde{P}(k_x = 0, k_y = 0) = P(k_x = 0, k_y = 0)$. They are

$$0, \quad -\frac{\sigma}{1-M_x} - \alpha, \quad -\frac{\sigma}{1+M_x} - \alpha, \quad -\frac{\sigma}{1-M_x^2} - \alpha.$$

The zero eigenvalue has multiplicity four and there could potentially be algebraic growth. However, straightforward calculations show that there are also four independent eigenvectors and this mode will be strongly stable.

For the case (b), the minimal polynomial of $\tilde{P}(ik)$ can be factored into

$$m_{\tilde{P}}(\lambda) = n_1(\lambda)n_2(\lambda)n_3(\lambda),$$
$$n_1(\lambda) = \lambda, \quad n_2(\lambda) = \lambda + \frac{\sigma + \alpha\zeta}{\zeta}, \quad n_3(\lambda) = m_2(\lambda; k_x = 0).$$

Directly, we see that the eigenvalues $\lambda = 0$ and $\lambda = -(\sigma/\zeta + \alpha)$, being solutions to $n_1(\lambda) = 0$ and $n_2(\lambda) = 0$, have non-positive real parts. The double zero eigenvalue of $\tilde{P}(ik)$ coresponds to the double eigenvalue $\lambda = -ik_yM_y$ of $P(ik)$. Associated with $\lambda = -ik_yM_y$, there are two linearly independent eigenvectors and thus stability will not be lost.

For $n_3(\lambda)$, we compute the coefficients in the continued fraction. They are

$$c_1 = -\frac{\zeta}{2(\sigma + \alpha\zeta)},$$
$$c_2 = -\frac{2(\sigma + \zeta\alpha)^2}{\sigma M_x^2 k_y^2 \zeta + (\sigma + \alpha(1 + M_x))(\sigma + \alpha(1 - M_x))(\sigma + \zeta\alpha)},$$
$$c_3 = -\frac{(\sigma M_x^2 k_y^2 \zeta + (\sigma + \alpha(1 + M_x))(\sigma + \alpha(1 - M_x))(\sigma + \zeta\alpha))^2}{2k_y^2 \zeta\sigma M_x^2(\sigma + \alpha)(\sigma\alpha + \zeta(\alpha^2 + k_y^2))},$$
$$c_4 = -\frac{2\sigma M_x^2(\sigma\alpha + \zeta(\alpha^2 + k_y^2))}{(\sigma + \alpha)(\sigma M_x^2 k_y^2 \zeta + (\sigma + \alpha(1 + M_x))(\sigma + \alpha(1 - M_x))(\sigma + \zeta\alpha))}.$$

Due to the assumptions $0 < M_x < 1$, $\zeta > 0$, $\sigma > 0$ and $\alpha > 0$ they are all negative.

Finally we consider case (c). For this case $m_{\tilde{P}}(\lambda)$ again factors into three polynomials

$$m_{\tilde{P}}(\lambda) = o_1(\lambda)o_2(\lambda)o_3(\lambda),$$
$$o_1(\lambda) = \lambda - ik_x\frac{\zeta}{M_x}, \quad o_2(\lambda) = m_1(\lambda),$$

$$o_3(\lambda) = \lambda^3 + \left(2\frac{\sigma + \zeta\alpha}{1 - M_x^2} + ik_x\frac{1 + M_x^2}{M_x}\right)\lambda^2 + \left(\frac{(\sigma + \alpha)^2 - \alpha^2 M_x^2}{\zeta} + ik_x\frac{2(\sigma + \alpha(1 + M_x^2))}{M_x}\right)\alpha + ik_x\frac{(\sigma + \alpha)^2 + \alpha^2 M_x^2}{M_x}.$$

The eigenvalue belonging to $o_1(\lambda)$ is distinct and does not affect strong stability and the polynomial $o_2(\lambda) = m_1(\lambda)$ has already been checked. It only remains to check the coefficients of the continued fraction arising from $o_3(\lambda)$. They are

$$c_1 = -\frac{\zeta}{2(\sigma + \alpha\zeta)}, \tag{80}$$

$$c_2 = -\frac{2(\sigma + \alpha\zeta)^3}{c_{2a}}, \tag{81}$$

$$c_3 = -\frac{c_{2a}^3 M_x^2}{4(\sigma + \alpha\zeta)^4 k_x^2 \sigma\zeta^2(\alpha^2 M_x^2 + (\sigma + \alpha)^2)} \times \frac{1}{\alpha M_x^2(\alpha + \sigma)(\sigma + \alpha\zeta)^2 + \zeta^2 k_x^2(\sigma + \alpha(1 + M_x^2))^2}, \tag{82}$$

where

$$c_{2a} = (\sigma+\alpha\zeta)^2((\sigma+\alpha)^2 - \alpha^2 M_x^2) + 2\sigma k_x^2\zeta^2(\sigma+\alpha(1+M_x^2)). \tag{83}$$

Clearly, (80)-(82) are negative and defined unless $k_x = k_y = 0$. However, that particular case has already been checked.

To see that (71) is a neccessary condition we use the parameterization $k_y = \kappa$, $k_x = \gamma\kappa$ and compute the minimal polynomial of $\tilde{P}(\kappa,\gamma)$ with μ as a free parameter. Again the minimal polynomial can be factored into a quadratic and a quartic. If we compute the coefficients in the continued fraction for the quartic, we see that for κ large the sign of the coefficients c_3 and c_4 will be determined by the sign of the expression

$$\kappa^4(M_x^2 - \gamma^2 + \gamma^2 M_x^2)(M_x^2\mu^2\gamma^2 + 2\gamma^2\mu M_x + \gamma^2 - \mu^2\gamma^2 - \mu^2). \tag{84}$$

Since the expression $(M_x^2 - \gamma^2 + \gamma^2 M_x^2)$ will change sign when $\gamma^2 = M_x^2/(1-M_x^2)$ we must choose μ such that the sign of the last expression in (84) changes simultaneously. Hence μ must satisfy

$$(M_x^4 - 1)\mu^2 + 2M_x^3\mu + M_x^2 = 0,$$

i.e. we must choose

$$\mu = -\frac{M_x}{1+M_x^2} \quad \text{or} \quad \mu = \frac{M_x}{1-M_x^2}.$$

The first choice will violate the conditions for c_2 when k_x is large and cannot be used, while the second choice, as we have seen above, yields a strongly stable PML.

We summarize the results in the following

LEMMA 13. *For constant $\sigma > 0$, $\alpha > 0$ and $0 < M_x < 1$, a necessary and sufficient condition for strong stability of the system (68) is that*

$$\mu = \frac{M_x}{1-M_x^2}. \tag{85}$$

6. A Stable PML for General 2×2 Symmetric Hyperbolic Systems. Our final example is the symmetric hyperbolic system

$$\frac{\partial u}{\partial t} + \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}}_{A} \frac{\partial u}{\partial x} + \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}}_{B} \frac{\partial u}{\partial y} = 0. \tag{86}$$

Here A and B are real matrices, and we can choose $a_{12} = 0$ without loss of generality. Note that the convective wave equation

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 u = C^2\nabla^2 u,$$

is a special case of (86), if we choose

$$a_{11} = M + C, \quad a_{22} = M - C, \quad b_{12} = C, \quad a_{12} = b_{11} = b_{22} = 0.$$

Equation (86) also contains the anisotropic wave equation as a special case:

$$\frac{\partial^2 u}{\partial t^2} = \nabla \cdot (T\,\nabla u),$$
$$T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a > 0, \quad c > 0, \quad ac - b^2 > 0,$$

describing electromagnetic waves propagating in an anisotropic dielectric media. Here

$$a_{11} = -a_{22}, \quad a_{12} = 0, \quad b_{11} = -b_{22},$$
$$a_{11} = \sqrt{a}, \quad b_{11} = \frac{b}{\sqrt{a}}, \quad b_{12} = \sqrt{c - \frac{b^2}{a}}.$$

The direction in which the waves supported by the system (86) propagate depends on the coefficients of A and B. If A is nonsingular there are three distinct cases

(i) $a_{11}a_{22} < 0, b_{12} \neq 0:$ Coupled waves moving in opposite x- directions.
(ii) $a_{11}a_{22} > 0, b_{12} \neq 0:$ Coupled waves moving in the same x- direction.
(iii) $b_{12} = 0,\ a_{11}a_{22} \neq 0:$ Decoupled waves.

For the cases (ii) and (iii), there is no need to use a PML since waves can be damped without reflection by simply adding a damping term

$$\begin{bmatrix} \sigma_1(x) & 0 \\ 0 & \sigma_2(x) \end{bmatrix} u,$$

to (86). The appropriate signs of σ_1 and σ_2 can be determined by the sign of a_{jj}; see also [4]. The case (i) is more interesting. In [4], the following PML model is suggested

$$\begin{aligned} \frac{\partial u}{\partial t} + A(\frac{\partial u}{\partial x} + \sigma\nu u + \phi) + B\frac{\partial u}{\partial y} = 0, \\ \frac{\partial \phi}{\partial t} + (\sigma+\alpha)\phi + \beta\frac{\partial \phi}{\partial y} = \sigma(\gamma\frac{\partial u}{\partial x} + \nu(\sigma+\alpha)u + \delta\frac{\partial u}{\partial y}), \end{aligned} \tag{87}$$

where

$$\delta = \frac{b_{22} - b_{11}}{a_{11} - a_{22}}, \quad \nu = -\frac{a_{11} + a_{22}}{2|a_{11}a_{22}|}, \tag{88}$$

$$\beta = -\frac{b_{11}a_{22} - b_{22}a_{11}}{a_{11} - a_{22}}, \quad \gamma = -1, \quad \alpha \geq 0. \tag{89}$$

LEMMA 14. *For $\sigma > 0$ and constant, $a_{11}a_{22} < 0$, and $\alpha \geq 0$ the system (87) is at least weakly stable.*

Proof. For simplicity we give the proof only for the case $\alpha = 0$. For the general choice of $\alpha \neq 0$ the coefficients are more complicated, but stability follows similarly. We consider three different cases $(a)\, k_x \neq 0, k_y \neq 0$, $(b)\, k_x \neq 0, k_y = 0$ and $(c)\, k_x = k_y = 0$. First we consider the case (a) and compute the coefficients in the continued fraction, (38) in Lemma 6. They are

$$c_1 = \frac{2a_{22}a_{11}}{\sigma\,(a_{11} - a_{22})^2},$$

$$c_2 = -\frac{2\sigma\left(a_{11}-a_{22}\right)^4}{c_{2a}},$$
$$c_{2a} = \sigma^2(a_{11}-a_{22})^4 + 4a_{11}^2a_{22}^2\left(k_x(a_{11}-a_{22}) + k_y(b_{11}-b_{22})\right)^2$$
$$-4a_{11}a_{22}b_{12}^2k_y^2(a_{11}+a_{22})^2\big),$$

$$c_3 = \frac{1}{32}\frac{c_{2a}^3}{\left(a_{11}-a_{22}\right)^2 c_{3a}a_{22}^2a_{11}^2b_{12}^2k_y^2\sigma},$$
$$c_{3a} = \sigma^2(a_{11}-a_{22})^4c_{3b} + 4a_{22}a_{11}c_{3b}^2,$$
$$c_{3b} = -k_y^2b_{12}^2(a_{11}+a_{22})^2 + a_{22}a_{11}(k_y(b_{11}-b_{22}) + k_x(a_{11}-a_{22}))^2,$$

$$c_4 = -8\frac{c_{3a}^3a_{22}a_{11}}{c_{4a}^4\sigma c_{4b}^2},$$

where c_{4a} and c_{4b} can be found in appendix C. The coefficients are negative except for the cases (b) and (c), but then the eigenvalues can be computed directly. For case (b) they are

$$0,\quad ika_{11}a_{22} - \frac{\sigma}{2}\left(1-\frac{a_{11}}{a_{22}}\right),\quad ika_{11}a_{22} - \frac{\sigma}{2}\left(1-\frac{a_{22}}{a_{11}}\right),$$

and for the case (c) they are

$$0,\quad -\frac{\sigma}{2}\left(1-\frac{a_{11}}{a_{22}}\right),\quad -\frac{\sigma}{2}\left(1-\frac{a_{22}}{a_{11}}\right).$$

Since $a_{11}a_{22} < 0$ they all have non negative real part, and the lemma is proved. □

As a final remark, in many cases the words "at least weakly", in Lemma 14, can be replaced by "strongly". However, to prove this we need to consider all cases when c_{4a} or c_{4b} vanish. Considering the complexity of the expressions c_{4a} and c_{4b} we expect the necessary calculations to be quite tedious.

7. Summary. We have presented a very general PML model for first order hyperbolic systems. We believe that the generality should make the model suitable for many future applications. We have also proven that the equations in the layer are perfectly matched to the equations in the computational domain. For the model formulated with one set of auxiliary variables, we have also showed that the layer equations always can be made strongly well-posed.

The critical step in the construction of a PML is to choose the free parameters so that the solution in the layer is stable. To simplify the analysis of this step, we have presented a method with which the stability of the layer can be determined by checking a fixed number of algebraic inequalities, which in turn can be generated automatically. Additionally, if these inequalities hold, we showed that there is an energy density in Fourier space that decays with time. By simple algebraic manipulations and application of Parseval's relation, this energy density can be converted to a decaying energy in physical space. The energy contains only the solution and its spatial and temporal derivatives; i.e. the energy is local.

We have used the introduced tecniques to show strong stability for a PML for Maxwell's equations and a PML for the linearized Euler equations. We also showed weak stability for a PML for a general 2×2 hyperbolic system in $(2+1)$ dimensions. For the PML for Maxwell's equations, we also derived a semi-local and a local energy. These energies guarantee the time-decay of higher order derivatives in space and time of the solution.

Unlike techniques that only involve checking the roots of the characteristic polynomial, our method is applicable to variable coefficient problems. This is important since in "real life" the damping parameter σ is not constant. The stability of the variable coefficient problem can be analyzed as a perturbation of the constant coefficient problem. If the constant coefficient problem is stable our method generates an energy. Since the energy decays for constant σ we expect it to decay at least for slowly varying σ.

Appendix A. The space-time energy version of $\mathcal{E}^{IV}$ is obtained by integration over all wavenumbers and application of Parseval. It is

$$
\begin{aligned}
E^{IV} = & \Bigg\| \Big(((\alpha - \varepsilon\partial_y^2 + \sigma)^3 - \sigma\partial_x^2)\partial_t(\frac{1}{2\sigma}(\alpha - \varepsilon\partial_y^2 + \sigma)\partial_t^2 + \frac{1}{\sigma}(\alpha - \varepsilon\partial_y^2 + \sigma)^2\partial_t + \frac{1}{2\sigma}) \\
& -\partial_x^2 \left((\alpha - \varepsilon\partial_y^2 + \sigma)((\alpha - \varepsilon\partial_y^2)^2 + \sigma(\alpha - \varepsilon\partial_y^2) - \partial_y^2) + \alpha - \varepsilon\partial_y^2 k_x^2 \right) \\
& \times(\frac{1}{2}\partial_t + \sigma + \alpha - \varepsilon\partial_y^2)\Big) v(\cdot, t) \Bigg\|^2 .
\end{aligned}
$$

For $\mathcal{E}^{III}$ we first rewrite

$$
\begin{aligned}
\frac{1}{2\sigma}(\tau+\sigma)((\tau+\sigma)^3 + \sigma k_x^2)\left((\tau+\sigma)^2 k_y^2 + k_x^2\tau^2\right) = \\
+\frac{1}{2\sigma}(\tau+\sigma)^6 k_y^2 + \frac{1}{2\sigma}(\tau+\sigma)^4 k_x^2\tau^2 \\
+\frac{\alpha+\sigma}{2}(\tau+\sigma)^2 k_x^2 k_y^2 + \frac{\alpha+\sigma}{2}\tau^2 k_x^4 + \frac{\varepsilon}{2}(\tau+\sigma)^2 k_x^2 k_y^4 + \frac{\varepsilon}{2}\tau^2 k_x^4 k_y^2.
\end{aligned}
$$

Integrating over k and applying Parseval to each term in $\mathcal{E}^{III}$ we get

$$
\begin{aligned}
E^{III} = & \frac{1}{2\sigma}\|(\alpha + \sigma - \varepsilon\partial_y^2)^3\partial_y \mathcal{F}^{-1}\{\hat{\chi}_2\}\|^2 \\
& + \frac{1}{2\sigma}\|(\alpha + \sigma - \varepsilon\partial_y^2)^2(\alpha - \varepsilon\partial_y^2)\partial_x \mathcal{F}^{-1}\{\hat{\chi}_2\}\|^2 \\
& + \frac{\sigma+\alpha}{2}\|(\alpha + \sigma - \varepsilon\partial_y^2)\partial_x\partial_y \mathcal{F}^{-1}\{\hat{\chi}_2\}\|^2 \\
& + \frac{\sigma+\alpha}{2}\|(\alpha - \varepsilon\partial_y^2)\partial_x^2 \mathcal{F}^{-1}\{\hat{\chi}_2\}\|^2 \\
& + \frac{\varepsilon}{2}\|(\alpha + \sigma - \varepsilon\partial_y^2)\partial_x\partial_y^2 \mathcal{F}^{-1}\{\hat{\chi}_2\}\|^2 \\
& + \frac{\varepsilon}{2}\|(\alpha - \varepsilon\partial_y^2)\partial_x^2\partial_y \mathcal{F}^{-1}\{\hat{\chi}_2\}\|^2,
\end{aligned}
$$

where

$$
\mathcal{F}^{-1}\{\hat{\chi}_2\} = \left((\sigma + \alpha - \varepsilon\partial_y^2)\partial_t^2 + 2(\sigma + \alpha - \varepsilon\partial_y^2)^2\partial_t + (\sigma + \alpha - \varepsilon\partial_y^2)^3 + \sigma\partial_x^2\right) v(x, t).
$$

In the same way we get for $\mathcal{E}^{II}$

$$
\begin{aligned}
E^{II} =& 2\alpha\|(\alpha+\sigma-\varepsilon\partial_y^2)^3\partial_x\partial_y\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\varepsilon\|(\alpha+\sigma-\varepsilon\partial_y^2)^3\partial_x\partial_y^2\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2(\alpha+\sigma)\|(\alpha+\sigma-\varepsilon\partial_y^2)^2\partial_x\partial_y^2\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\varepsilon\|(\alpha+\sigma-\varepsilon\partial_y^2)^2\partial_x\partial_y^3\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\alpha\|(\alpha+\sigma-\varepsilon\partial_y^2)^2\partial_x^2\partial_y\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\varepsilon\|(\alpha+\sigma-\varepsilon\partial_y^2)^2\partial_x^2\partial_y^2\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\alpha\|(\alpha+\sigma-\varepsilon\partial_y^2)^2(\alpha-\varepsilon\partial_y^2)\partial_x^2\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\varepsilon\|(\alpha+\sigma-\varepsilon\partial_y^2)^2(\alpha-\varepsilon\partial_y^2)\partial_x^2\partial_y\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2(\alpha+\sigma)\|(\alpha+\sigma-\varepsilon\partial_y^2)(\alpha-\varepsilon\partial_y^2)\partial_x\partial_y\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\varepsilon\|(\alpha+\sigma-\varepsilon\partial_y^2)(\alpha-\varepsilon\partial_y^2)\partial_x^2\partial_y^2\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\alpha\|(\alpha+\sigma-\varepsilon\partial_y^2)(\alpha-\varepsilon\partial_y^2)\partial_x^3\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2 \\
&+2\varepsilon\|(\alpha+\sigma-\varepsilon\partial_y^2)(\alpha-\varepsilon\partial_y^2)\partial_x^3\partial_y\mathcal{F}^{-1}\{\hat{\chi}_1\}\|^2,
\end{aligned}
$$

where

$$
\mathcal{F}^{-1}\{\hat{\chi}_1\} = (\frac{1}{2}\partial_t+\sigma+\alpha-\varepsilon\partial_y^2)v \tag{90}
$$

By similar operations, we can obtain an expression for E^I. However, since we have to split the the factor in front of $|\hat{v}|^2$ in $\mathcal{E}^I$ in many terms the expression for E^I becomes very lengthy and we have choosen not to include it here.

Appendix B.

$$
\begin{aligned}
c_{2a} =& \sigma\zeta^2(\sigma+3M_x^2\alpha+\alpha)k_x^2+M_x^2\sigma\zeta(\sigma+\zeta\alpha)k_y^2 \\
&+(\sigma+\zeta\alpha)^2(\sigma+\alpha(1+M_x))(\sigma+\alpha(1-M_x)),
\end{aligned}
$$

$$
\begin{aligned}
c_{3a} &= (c_{3b}+c_{3c}k_x^4+c_{3d}k_y^4+c_{3e}k_x^2k_y^2+c_{3f}k_x^2+c_{3g}k_y^2), \\
c_{3b} &= (-\zeta k_x^2+M_x^2k_y^2)^2(\sigma\alpha\zeta^3k_x^2+\sigma(\sigma+\alpha)\zeta^2k_y^2), \\
c_{3c} &= \alpha\zeta^3(5{M_x}^4\alpha^3+12\sigma{M_x}^2\alpha^2+10{M_x}^2\alpha^3+2\alpha\sigma^2{M_x}^2+5\alpha\sigma^2+ \\
&\quad 2\sigma^3+4\sigma\alpha^2+\alpha^3),
\end{aligned}
$$

$$
\begin{aligned}
c_{3d} &= M_x^2(\sigma+\alpha)\zeta(\alpha\zeta+\sigma)(\zeta\alpha^2+2\sigma\alpha+\sigma^2+\sigma M_x^2\alpha), \\
c_{3e} &= -\zeta^2(\sigma^4+2\sigma^3\alpha(2+M_x^2)+\sigma^2\alpha^2(6+11M_x^2+M_x^4) \\
&\quad +4\sigma\alpha^3(1+4M_x^2-M_x^6)+\alpha\eta(1+8M_x^2+3M_x^4)), \\
c_{3f} &= \alpha\zeta(\sigma+\alpha)(3\alpha^2M_x^2+(\alpha+\sigma)^2)(\alpha\zeta+\sigma)^2, \\
c_{3g} &= \alpha M_x^2(\sigma+\alpha)(\sigma+\alpha(1+M_x))(\sigma+\alpha(1-M_x))(\alpha\zeta+\sigma)^2,
\end{aligned}
$$

$$
\begin{aligned}
c_{4a} &= \alpha^2(\sigma+\alpha)^2(\zeta\alpha+\sigma)^2+c_{4b}k_x^4+c_{4c}k_y^4+c_{4d}k_x^2k_y^2+c_{4e}k_x^2+c_{4f}k_y^2, \\
c_{4b} &= \zeta^4\alpha^2, \quad c_{4c}=\zeta^2(\sigma+\alpha)^2, \quad c_{4d}=2\zeta^3\alpha(\sigma+\alpha), \\
c_{4e} &= 2\alpha^2(\sigma+\alpha)\zeta^2\left(\alpha(1+{M_x}^2)+\sigma\right), \\
c_{4f} &= 2\alpha\zeta(\sigma+\alpha)^2(\alpha\zeta+\sigma).
\end{aligned}
$$

Appendix C.

$$c_{4a} = (4{k_x}^2{a_{11}}^4{a_{22}}^2 + \sigma^2{a_{11}}^4 - 4{k_y}^2{b_{12}}^2{a_{11}}^3a_{22} - 8k_yb_{22}{a_{11}}^3k_x{a_{22}}^2$$
$$+8k_x{a_{11}}^3{a_{22}}^2k_yb_{11} - 4\sigma^2{a_{11}}^3a_{22} - 8{k_x}^2{a_{11}}^3{a_{22}}^3 + 6\sigma^2{a_{22}}^2{a_{11}}^2$$
$$+4{k_x}^2{a_{11}}^2{a_{22}}^4 - 8{k_y}^2b_{22}{a_{11}}^2b_{11}{a_{22}}^2 - 8{k_y}^2{b_{12}}^2{a_{11}}^2{a_{22}}^2 + 8k_x{a_{11}}^2{a_{22}}^3k_yb_{22}$$
$$+4{k_y}^2{b_{11}}^2{a_{22}}^2{a_{11}}^2 + 4{k_y}^2{b_{22}}^2{a_{11}}^2{a_{22}}^2 - 8k_x{a_{11}}^2{a_{22}}^3k_yb_{11} - 4{k_y}^2{b_{12}}^2a_{11}{a_{22}}^3$$
$$-4\sigma^2{a_{22}}^3a_{11} + \sigma^2{a_{22}}^4),$$
$$c_{4b} = ({a_{22}}^2{b_{12}}^2{k_y}^2 + {b_{11}}^2a_{22}a_{11}{k_y}^2 + 2a_{22}a_{11}{k_y}^2{b_{12}}^2 - 2b_{11}a_{22}a_{11}{k_y}^2b_{22}$$
$$+a_{22}a_{11}{k_y}^2{b_{22}}^2 + {b_{12}}^2{k_y}^2{a_{11}}^2 - 2b_{11}k_ya_{11}{a_{22}}^2k_x + 2k_ya_{11}{a_{22}}^2k_xb_{22}$$
$$+2b_{11}k_y{a_{11}}^2a_{22}k_x - 2k_y{a_{11}}^2a_{22}k_xb_{22} + {a_{22}}^3a_{11}{k_x}^2 - 2{a_{11}}^2{a_{22}}^2{k_x}^2 + a_{22}{a_{11}}^3{k_x}^2).$$

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