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Automatic Symmetrization and Energy Estimates Using Local Operators for Partial Differential Equations

THOMAS HAGSTROM[1] AND DANIEL APPELÖ[2]

[1]Department of Mathematics and Statistics,
The University of New Mexico, Albuquerque, New Mexico, USA
[2]Department of Numerical Analysis and Computer Science,
Royal Institute of Technology, Stockholm, Sweden

We develop a method for automatically symmetrizing Petrowsky well-posed Cauchy problems for constant coefficient linear partial differential equations. The method is rooted in the Sturm sequence technique for establishing the location of the roots of a complex polynomial and can be automated using standard symbolic computation tools. In the special case of homogeneous strictly hyperbolic scalar equations, we show that the resulting estimates are strong enough to control all principal order derivatives and thus can be used in place of the Leray energies. We also illustrate the method by applying it to various problems of mixed type.

Keywords Cauchy problem; Energy estimates; Sturm sequences; Well-posedness.

Mathematics Subject Classification Primary 35E15; Secondary 35B35.

1. Introduction

The basic issue in the analysis of the Cauchy problem for partial differential equations is the determination of well-posedness, weak or strong. In the constant coefficient case this question can be shown to be equivalent to certain algebraic conditions on the ordinary differential operators in time obtained by Fourier transformation in space. For strong well-posedness we require estimates of Sobolev norms of the solution in terms of the same Sobolev norms of the Cauchy data. The algebraic theory is summarized in the Kreiss Matrix Theorem which establishes necessary and sufficient conditions for the uniform boundedness of exponentials

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Address correspondence to Thomas Hagstrom, Department of Mathematics and Statistics, MSC 03 2150, 1 University of New Mexico, Albuquerque, NM 87131, USA; E-mail: hagstrom@math.unm.edu

of families of matrices. These conditions involve stronger requirements than simple upper bounds on the real parts of the exponents appearing in exponential solutions of the transformed problem; see Kreiss and Lorenz (1989, Ch. 2). Weak well-posedness or well-posedness in the sense of Petrowsky, requires the weaker condition that the norms of the solution can be estimated by some possibly stronger Sobolev norm of the data. As shown, e.g., in Gindikin and Volevich (1991), a consequence of the Seidenberg-Tarski Theorem is that Petrowsky well-posedness can be established simply by deriving a bound on the real parts of the exponents, which in turn follows from establishing uniform bounds on the real parts of the roots of a family of polynomials.

Extensions of the theory to variable coefficient and nonlinear problems often rely on the construction of symmetrizers and energy estimates for related constant coefficient systems. The Kreiss Matrix Theorem establishes the existence of such a symmetrizer in the case of strongly well-posed problems. However, the construction is not direct in that it depends on the Schur form of the symbol, which can not be computed in general. Moreover, the resulting energies typically involve nonlocal operators, though localizable symmetrizers can be found (Kreiss, 1963). In this work we use the method of Sturm sequences for bounding the real parts of roots of polynomials, which by the remarks above provide an automatic technique for checking Petrowsky well-posedness of a general Cauchy problem. We prove that this method can be adapted to automatically symmetrize any well-posed problem, producing an energy estimate which is localizable; that is an energy which, after application of a differential operator, can be expressed in terms of integrals of squares of functions obtained by applying certain other differential operators to solution components. The energy density itself can be directly computed by symbolic means.

The remainder of the paper is organized as follows. In Section 2 we recall the Sturm sequence construction as based on the Euclidean division algorithm. In Section 3 we carry out the basic symmetrizer construction for scalar ordinary differential equations, commenting on the extension to systems in Section 4. In Section 5 we apply the symmetrization to partial differential equations and show how to localize the resulting energy density. Section 6 contains a number of examples including problems of hyperbolic and hyperbolic-parabolic type as well as a Boussinesq system with high-order spatial derivatives. Finally, we conclude in Section 7, mentioning some open questions associated with this technique.

2. Mathematical Preliminaries

Determinant criteria for computing the number of zeros of a polynomial lying in a half-plane are well-known and extensively reviewed in Marden (1949). The following simple algorithm, producing the so-called Sturm sequence, is equivalent to the determinant conditions and will serve as the basic ingredient in the construction of a symmetrizer.

Let $P(x)$ be a complex polynomial such that with the substitution $x = iz$:

$$P(iz) = P_0(z) + iP_1(z), \tag{1}$$

where P_0 is a real monic[1] polynomial of degree n and P_1 a real polynomial of degree $m < n$. Suppose further that P_1 is not identically zero; if it is identically zero P

[1]A polynomial is called monic if its lead coefficient is one.

obviously can not have all its roots satisfying $\Re x < 0$. Using the standard division algorithm, form the sequence of real polynomials:

$$P_{k-1}(z) = Q_k(z)P_k(z) - P_{k+1}(z), \quad k = 1, \ldots, \mu - 1, \tag{2}$$

$$\deg Q_k = \deg P_{k-1} - \deg P_k > 0, \tag{3}$$

terminated when $P_{\mu+1}(z) \equiv 0$. We note that $\mu \leq n$. We then have the following theorem, which is a special case of Marden (1949, Thm. 38,1). (See also Wall, 1948, Thm. 47.1.)

Theorem 1. *All zeros of $P(x)$ satisfy $\Re x < 0$ if and only if $\mu = n$ and the lead coefficients, $-\lambda_k^2$, of the linear polynomials Q_k, are negative.*

3. Symmetrization

We now use Theorem 1 to prove our main theorem. Consider the scalar ordinary differential equation:

$$R\left(\frac{d}{dt}\right)y = 0, \tag{4}$$

where R is a degree n polynomial. Noting that for some $\alpha > 0$, $e^{-\alpha t}y$ decays as $t \to \infty$, we define P by:

$$P\left(\frac{d}{dt}\right) = \alpha_0 R\left(\frac{d}{dt} + \alpha\right), \tag{5}$$

where the complex number α_0 is chosen so that P_0 is monic. Forming the Sturm sequence, we have that:

$$Q_k(z) = -\lambda_k^2 z + \eta_k. \tag{6}$$

Also, we note that if the polynomials P and P_0 had common roots then P_0 and P_1 would also share those common roots. In this case the degree of P_2 would be less than $n - 2$, contradicting Theorem 1. Thus

$$v = e^{-\alpha t}y, \tag{7}$$

can be written as:

$$v = P_0\left(-i\frac{d}{dt}\right)w, \tag{8}$$

where

$$P\left(\frac{d}{dt}\right)w = 0. \tag{9}$$

Set:

$$v^{(k)} = P_k\left(-i\frac{d}{dt}\right)w. \tag{10}$$

Then by (2) the functions $v^{(k)}$ satisfy:

$$\lambda_k^2 \frac{dv^{(k)}}{dt} = i\big(\eta_k v^{(k)} - v^{(k-1)} - v^{(k+1)}\big), \quad k = 1, \ldots, n, \tag{11}$$

where $v^{(n+1)} = 0$, and by (8), (9), and (10) with $k = 1$:

$$v \equiv v^{(0)} = -iv^{(1)}. \tag{12}$$

In matrix form (11) becomes:

$$\Lambda^2 \frac{dV}{dt} = HV, \tag{13}$$

where

$$H + H^* = -2e_1 e_1^T, \quad V = (v^{(1)}, \ldots, v^{(n)})^T, \tag{14}$$

(Here e_1 is the standard unit n-vector). Using (13) and (12) standard computations yield:

Theorem 2. *The functions $v^{(k)}$ satisfy the energy equality*:

$$\frac{dE}{dt} = -2|v|^2, \tag{15}$$

where

$$E = \sum_{k=1}^{n} \lambda_k^2 |v^{(k)}|^2.$$

Remark. We note that the construction of the symmetrized system (13) and associated energy via (2) is direct and that the coefficients are rationally dependent on the coefficients of the original equation, R. No roots of polynomials are required. Thus it is straightforward to carry out the process explicitly by symbolic computations, even for problems depending on parameters such as those arising from the Fourier transformation in space of constant coefficient partial differential equations. We have used symbolic computations to derive the energies in some of the examples below as well as the energies for the Perfectly Matched Layer (PML) models studied in Appelö et al. (2006).

The energy can be directly expressed in terms of the function, v, by solving recursively (12) and (11):

$$v^{(1)} = iv, \quad v^{(k+1)} = i\lambda_k^2 \frac{dv^{(k)}}{dt} + \eta_k v^{(k)} - v^{(k-1)} \equiv L_{k+1}\left(-i\frac{d}{dt}\right)v. \tag{16}$$

Clearly the polynomials L_k satisfy the recursion:

$$L_{k+1}(z) = Q_k(z)L_k(z) - L_{k-1}(z), \tag{17}$$

where $L_0 = 1$, $L_1 = i$.

To establish an optimal decay rate we must simply find the greatest lower bound of $\Re\alpha$ from the set of all α leading to $-\lambda_k^2 < 0$.

3.1. *A Simple Example*

To illustrate the construction above, consider the simple case of a damped harmonic oscillator:

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega^2 y = 0. \tag{18}$$

Then:

$$\begin{aligned} P\left(\frac{d}{dt}\right) &= \left(-i\frac{d}{dt}\right)^2 - i(2\alpha+\gamma)\left(-i\frac{d}{dt}\right) - (\alpha^2+\alpha\gamma+\omega^2) \\ &= z^2 - i(2\alpha+\gamma)z - (\alpha^2+\alpha\gamma+\omega^2). \end{aligned} \tag{19}$$

Thus:

$$P_0(z) = z^2 - (\alpha^2+\alpha\gamma+\omega^2), \quad P_1(z) = -(2\alpha+\gamma)z. \tag{20}$$

Applying the division algorithm we find:

$$Q_1(z) = -\frac{1}{2\alpha+\gamma}z, \quad Q_2(z) = -\frac{2\alpha+\gamma}{\alpha^2+\alpha\gamma+\omega^2}z. \tag{21}$$

For positive damping, $\gamma > 0$, we clearly have $\lambda_{1,2}^2 > 0$ with $\alpha = 0$, which we now impose. Then the energy is given by:

$$E = \frac{1}{\gamma}y^2 + \frac{\gamma}{\omega^2}\left(\frac{1}{\gamma}\frac{dy}{dt} + y\right)^2, \tag{22}$$

and satisfies

$$\frac{dE}{dt} = -2y^2. \tag{23}$$

4. Application to Systems

For systems of equations, the most straightforward approach is to derive scalar equations for each component, and then derive the energy equality above for each scalar equation. For example, for the first order system:

$$\frac{du}{dt} = Au, \quad u \in \mathbb{R}^d, \ A \in \mathbb{R}^{d\times d}, \tag{24}$$

we define the minimal polynomial of A to be the monic polynomial, P, of least degree such that:

$$P(A) = 0. \tag{25}$$

(Since the Cayley–Hamilton Theorem implies that A satisfies the characteristic equation, the degree of P is at most d.)

A simple induction argument then shows that each component, u_i, satisfies:

$$P\left(\frac{d}{dt}\right)u_i = 0. \tag{26}$$

Thus each component satisfies the energy equality constructed from P. We note that the use of the minimal polynomial rather than the characteristic polynomial is crucial to the construction of optimal estimates for hyperbolic problems as it allows us to distinguish between diagonalizable and nondiagonalizable systems with multiple characteristics. See the examples below.

A disadvantage of the construction based on the minimal polynomial is that we derive the same equation for each component. A more general technique, which can be applied to the lth-order system:

$$\sum_{j=0}^{l} R_j \frac{d^j u}{dt^j} = 0, \quad u \in \mathbb{R}^d, \;\; R_j \in \mathbb{R}^{d\times d}, \tag{27}$$

is the Smith factorization (Wloka et al., 1995, Ch. 1). The Smith factorization leads to new variables, $\tilde{u}_i$, which satisfy scalar equations of increasing order:

$$D_i\left(\frac{d}{dt}\right)\tilde{u}_i = 0. \tag{28}$$

Here D_i is a monic polynomial which divides D_{i+1}. Again, the energy construction can be applied to each of the equations (28). As the differential operators D_i appearing in the Smith factorization are of minimal order for the underlying system we suspect that their use will lead to optimal estimates. However, in our examples we have not yet implemented an automatic Smith factorization algorithm, and so our energies for systems were constructed via the minimal polynomial. Then we have used the governing equations themselves to refine the results.

5. Application to PDEs

Consider the scalar, constant coefficient partial differential equation:

$$R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)u = 0, \quad (t, x) \subset \mathbb{R} \times \mathbb{R}^m, \tag{29}$$

where R is now a polynomial in $m+1$ variables. (Note that by the discussion above the construction can also be applied to systems.) Performing a Fourier transformation in the space variables we obtain:

$$\widehat{R}\left(\frac{d}{dt}, i\zeta_1, \ldots, i\zeta_m\right)\hat{u} = 0. \tag{30}$$

We now suppose that the Cauchy problem in t defined by (29) is well-posed in the sense of Petrowsky. Then (see Gindikin and Volevich, 1991, Ch. 1) there exists

M_0 independent of $\zeta \in \mathbb{R}^m$ such that $\hat{v} = e^{-\alpha t}\hat{u}$ is exponentially decaying for all α with $\Re\alpha > M_0$. Choosing such an α and defining $\widehat{P}$ by:

$$\widehat{P} = \hat{\alpha}_0 \widehat{R}\left(\frac{d}{dt} + \alpha, i\zeta_1, \ldots, i\zeta_m\right), \tag{31}$$

we construct the symmetrized system:

$$\widehat{\Lambda}^2 \frac{d\widehat{V}}{dt} = \widehat{H}\widehat{V}, \tag{32}$$

and energy functional $\widehat{E}$ as above. We note that all coefficients in the definitions of the functions $\hat{v}^{(k)}$ as well as the parameters $\hat{\lambda}_k^2$ in the definition of $\widehat{E}$ are rational functions of ζ. Thus by multiplying the energy equality by a polynomial in ζ, integrating over ζ, and invoking Parseval's equality for the L_2 norm, $\|\cdot\|$, we prove:

Theorem 3. *Suppose* (29) *is well-posed in the sense of Petrowsky and let* $\Re\alpha$ *be sufficiently large. Then there exist nonzero partial differential operators* $\Phi_k\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}, \alpha\right)$, $k = 1, \ldots, S$ *and* $\Psi\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}, \alpha\right)$ *such that*:

$$\frac{dE}{dt} = -2\|\Psi v\|^2,$$

where

$$E = \sum_{k=1}^{S} \|\Phi_k v\|^2,$$

and $v = e^{-\alpha t}u$. *The coefficients of the differential operators,* Φ_k *and* Ψ, *can be directly computed from the polynomials in the Sturm sequence.*

Proof. Explicitly eliminating the $\hat{v}^{(k)}$, the energy equality guaranteed by Theorem 2 takes the form:

$$\frac{d}{dt}\left(\sum_{k=1}^{n} \hat{\lambda}_k^2 \left|\mathscr{L}_k\left(\frac{d}{dt}, \zeta\right)\hat{v}\right|^2\right) = -2|\hat{v}|^2, \tag{33}$$

where $\mathscr{L}_1 = 1$. Since by construction $\hat{\lambda}_k^2$ and the coefficients of the operators $\mathscr{L}_k$ are rational functions of ζ, there exists a real polynomial, $\psi(\zeta)$, such that:

$$\psi^2 \hat{\lambda}_k^2 = \mu_k^2, \quad \psi \mathscr{L}_k = \mathscr{M}_k, \tag{34}$$

where μ_k^2 and the coefficients of $\mathscr{M}_k$ are real polynomials in ζ. (All functions here also depend on α.) Multiplying (33) by ψ^4

$$\frac{d}{dt}\left(\sum_{k=1}^{n} \mu_k^2 \left|\mathscr{M}_k\left(\frac{d}{dt}, \zeta\right)\hat{v}\right|^2\right) = -2|\psi^2\hat{v}|^2. \tag{35}$$

Since $\mu_k^2 > 0$, by Hilbert's Seventeenth Problem ψ can be chosen so that (Reznick, 2000):

$$\mu_k^2 = \sum_{l=1}^{d_k} \phi_{lk}^2, \tag{36}$$

where ϕ_{lk} is a polynomial in ζ. We now may define the differential operators appearing in the statement of the theorem via their symbols:

$$\sum_{k=1}^{n} \sum_{l=1}^{d_k} |\phi_{lk} \mathcal{M}_k \hat{v}|^2 = \sum_{k=1}^{S} |\widehat{\Phi}_k \hat{v}|^2, \tag{37}$$

$$|\psi^2 \hat{v}|^2 = |\widehat{\Psi} \hat{v}|^2. \tag{38}$$

The theorem now follows from Parseval's equality. □

We remark that the determination of α, and for that matter the existence of an appropriate α, is a decidable problem to which known algorithms can be applied. See the articles in Caviness and Johnson (1998) for a discussion.

6. Examples

6.1. *Hyperbolic Equations*

We first consider scalar hyperbolic equations. For problems which are strictly hyperbolic, that is which have no multiple characteristics, general energy estimates were developed by Leray and used to prove the strong well-posedness of variable coefficient systems. See, for example, Gårding (1957). We also note that the Leray construction can be generalized to other Petrowsky well-posed systems as shown in Gindikin and Volevich (1991, Ch. 3). Here we will show that in the strictly hyperbolic case our construction also produces an energy for which strong well-posedness is evident; that is we produce an energy which controls all derivatives of principal order. In addition, we consider examples due to Peyser (1963) whose principal parts are not strictly hyperbolic.

6.1.1. The Convective Wave Equation. We begin with a simple example for which the Sturm sequence can easily be computed by hand. Consider the convective wave equation:

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 u = c^2 \nabla^2 u. \tag{39}$$

Note that we recover the classical wave equation by setting $M = 0$. Choose $\alpha > 0$, real. We then have:

$$\widehat{P}\left(\frac{d}{dt}\right) = \left(-i\frac{d}{dt} - i\alpha + \zeta_1 M\right)^2 - c^2|\zeta|^2. \tag{40}$$

Thus:

$$\widehat{P}_0(z) = z^2 + 2\zeta_1 M z - c^2|\zeta|^2 + M^2\zeta_1^2 - \alpha^2, \tag{41}$$

$$\widehat{P}_1(z) = -2\alpha z - 2\alpha M\zeta_1. \tag{42}$$

We now compute the Sturm sequence:

$$\widehat{Q}_1(z) = -\frac{z}{2\alpha} - \frac{M\zeta_1}{2\alpha}, \quad \widehat{P}_2(z) = c^2|\zeta|^2 + \alpha^2, \tag{43}$$

$$\widehat{Q}_2(z) = -\frac{2\alpha z}{c^2|\zeta|^2 + \alpha^2} - \frac{2\alpha M\zeta_1}{c^2|\zeta|^2 + \alpha^2}. \tag{44}$$

We see that any $\alpha > 0$ suffices. We have:

$$\hat{\lambda}_1^2 = \frac{1}{2\alpha}, \quad \hat{\lambda}_2^2 = \frac{2\alpha}{c^2|\zeta|^2 + \alpha^2}, \tag{45}$$

$$\hat{v}^{(1)} = i\hat{v}, \quad \hat{v}^{(2)} = -\frac{1}{2\alpha}\left(\frac{d\hat{v}}{dt} + iM\zeta_1\hat{v} + 2\alpha\hat{v}\right). \tag{46}$$

Thus the energy equality of Theorem 2 is:

$$\frac{1}{2\alpha}\frac{d}{dt}\left(|\hat{v}|^2 + \frac{1}{c^2|\zeta|^2 + \alpha^2}\left|\frac{d\hat{v}}{dt} + iM\zeta_1\hat{v} + 2\alpha\hat{v}\right|^2\right) = -2|\hat{v}|^2. \tag{47}$$

Clearing denominators and applying Parseval's relation we find for any $\alpha > 0$:

$$\frac{d}{dt}\int_{\mathbb{R}^m}\left(c^2|\nabla v|^2 + \alpha^2|v|^2 + \left|\frac{\partial v}{\partial t} + M\frac{\partial v}{\partial x} + 2\alpha v\right|^2\right)dx = -4\alpha\int_{\mathbb{R}^m}\left(c^2|\nabla v|^2 + \alpha^2|v|^2\right), \tag{48}$$

or letting $\alpha \to 0$:

$$\frac{d}{dt}\int_{\mathbb{R}^m}\left(c^2|\nabla u|^2 + \left|\frac{\partial u}{\partial t} + M\frac{\partial u}{\partial x}\right|^2\right)dx = 0. \tag{49}$$

This is the standard energy for the convective wave equation, or for the wave equation if $M = 0$. We note that using (48) we may conclude that the Cauchy problem for (39) is strongly well-posed.

6.1.2. A Second Order Problem with Multiple Characteristics. Now consider the problem:

$$\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 u = 0. \tag{50}$$

The calculations are identical to those presented above; one simply sets $c = 0$ in the formulas. However, it is clear that the resulting energy inequality can not be used to control all first derivatives:

$$\frac{d}{dt}\int_{\mathbb{R}^m}\left(\alpha^2|v|^2+\left|\frac{\partial v}{\partial t}+M\frac{\partial v}{\partial x}+2\alpha v\right|^2\right)dx=-4\alpha^3\int_{\mathbb{R}^m}|v|^2. \tag{51}$$

This phenomenon occurs whenever we have multiple characteristics.

We note that for diagonalizable systems with multiple characteristics of constant multiplicity, use of the minimal polynomial or Smith factorization will lead to scalar systems which are strictly hyperbolic. Thus, as we will see below, our energy will produce estimates which establish strong well-posedness.

6.1.3. Strictly Hyperbolic Scalar Equations of Higher Order. Suppose now that the polynomial $\widehat{R}$ is homogeneous in d/dt and ζ and of degree n. Further suppose it is strictly hyperbolic; that is for all ζ we may factor $\widehat{P}$:

$$\widehat{P}(iz)=\prod_{j=1}^{n}(z-c_j(\zeta)-i\alpha), \tag{52}$$

where the wave speeds c_j are real and distinct. Take $\alpha \ll 1$ and write:

$$\widehat{P}=\widetilde{P}_0-i\alpha\widetilde{P}_0'+O(\alpha^2), \tag{53}$$

where prime denotes differentiation with respect to z and

$$\widetilde{P}_0=\prod_{j=1}^{n}(z-c_j(\zeta)). \tag{54}$$

We apply the algorithm (2) to the pair $\widetilde{P}_0$, $\widetilde{P}_1=-\widetilde{P}_0'$, denoting by tilde the polynomials thus produced. We then have:

Lemma 1. *All roots of $\widetilde{P}_0$ are real and distinct if and only if $\mu=n$ and the lead coefficients, $-\widetilde{\lambda}_k^2$, of the linear polynomials $\widetilde{Q}_k$, are negative.*

Proof. Since

$$\frac{\widetilde{P}_0'}{\widetilde{P}_0}=\sum_{j=1}^{n}\frac{1}{z-c_j(\zeta)}, \tag{55}$$

the lemma is a direct consequence of Wall (1948, Thm. 43.1). □

Fix $|\zeta|=1$. Noting that:

$$\widehat{P}_0=\widetilde{P}_0+O(\alpha^2),\quad \widehat{P}_1=-\alpha\widetilde{P}_0'+O(\alpha^3) \tag{56}$$

we rescale (2) by replacing $\widehat{P}_1$ by $\overline{P}_1=\alpha^{-1}\widehat{P}_1$ and similarly compute the rescaled auxiliary variables $\bar{v}^{(k)}$. That is, we define $\bar{v}^{(k)}=\alpha^{-1}\hat{v}^{(k)}$. Then the barred variables are defined through polynomials $\overline{L}_k$ satisfying:

$$\overline{L}_{k+1}(z)=\overline{Q}_k(z)\overline{L}_k(z)-\overline{L}_{k-1}(z), \tag{57}$$

where $\overline{L}_0 = \alpha$, $\overline{L}_1 = i$. This leads to a rescaled energy equality involving the barred variables:

$$\frac{d\overline{E}}{dt} = -2\alpha^2|\bar{v}|^2. \tag{58}$$

By Lemma 1 and the homogeneity of R we conclude that:

$$\bar{\lambda}_k^2 = \tilde{\lambda}_k^2 + O(\alpha^2), \quad \bar{\eta}_k = \tilde{\eta}_k + O(\alpha^2), \tag{59}$$

uniformly. Taking the limit $\alpha \to 0$ we obtain:

$$\frac{d\overline{E}_0}{dt} = 0, \tag{60}$$

where $\overline{E}_0$ is determined by the Sturm sequence defined by $\widetilde{P}_0$ and $\widetilde{P}_0'$. Since $\widetilde{P}_0$ is homogeneous of degree n and $\widetilde{P}_0'$ is homogeneous of degree $n-1$ we can track the homogeneity properties of all terms in the sequence via (2). In particular we find that the homogeneity of the $\widetilde{P}_k$ alternates between n and $n-1$ and thus the homogeneity of $\tilde{\lambda}_k^2$ alternates between 0 and -2. Moreover, we have that $\overline{L}_k = \widetilde{L}_k + O(\alpha)$ where:

$$\widetilde{L}_{k+1}(z) = \widetilde{Q}_k(z)\widetilde{L}_k(z) - \widetilde{L}_{k-1}(z), \tag{61}$$

with $\widetilde{L}_0 = 0$, $\widetilde{L}_1 = i$. From these relations we conclude that $\tilde{\lambda}_k^2|\widetilde{L}_k|^2$, is homogeneous of degree 0. Moreover, since the polynomial $\widetilde{L}_k$ is of exact degree $k-1$ in z the collection forms a basis for the degree $n-1$ polynomials in z for each ζ. Thus

$$\overline{E}_0 = \sum_{k=1}^{n} \tilde{\lambda}_k^2|\widetilde{L}_k|^2|\hat{v}|^2, \tag{62}$$

and using the compactness of the unit sphere we conclude that for some $c > 0$:

$$\overline{E}_0 \geq c\sum_{j=0}^{n-1} |z|^{2j}|\hat{v}|^2. \tag{63}$$

Multiplying by $|\zeta|^{2n-2}$ and using homogeneity and Parseval's relation we finally conclude:

Theorem 4. *Let E_0 be the $\alpha \to 0$ limit of the energy determined by the Sturm sequence for a homogeneous nth order strictly hyperbolic equation scaled so that its symbol is homogeneous of degree $2n-2$. Then there exists a constant c such that*:

$$E_0 \geq c\sum_{j=0}^{n-1} \sum_{|v|=n-1-j} \left\| D^v \frac{\partial^j u}{\partial t^j} \right\|^2.$$

We note, as in Gårding (1957), that this estimate directly implies the well-posedness of the system resulting from lower order perturbations and establishes the strong well-posedness of the Cauchy problem.

6.1.4. Peyser's Examples. Applying the previous construction to homogeneous hyperbolic problems with multiple characteristics leads to energies, E_0, which do not control all $n-1$st order derivatives. As a result, the Cauchy problem becomes ill-posed with certain lower order perturbations. Peyser (1963) has identified classes of lower order perturbations of equations whose principal parts have multiple characteristics that still produce a well-posed problem. Obviously our construction also discriminates between lower-order terms which destabilize the Cauchy problem and those which do not. Here we consider the examples from Peyser (1963) to illustrate this point.

The principal part of Peyser's examples in $\mathbb{R}^2$ is:

$$P_{Pey} = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{\partial^2}{\partial x_2^2}\right). \tag{64}$$

We begin by constructing the Sturm sequence associated with the principal part. The coefficients λ_k^2 are then given by:

$$\lambda_1^2 = \frac{1}{4\alpha}, \quad \lambda_2^2 = \frac{8\alpha}{10\alpha^2 + 2\zeta_1^2 + 3\zeta_2^2}, \tag{65}$$

$$\lambda_3^2 = \frac{(10\alpha^2 + 2\zeta_1^2 + 3\zeta_2^2)^2}{4\alpha(\zeta_2^4 + 16\alpha^2\zeta_1^2 + 24\alpha^2\zeta_2^2 + 16\alpha^4)}, \tag{66}$$

$$\lambda_4^2 = \frac{2\alpha(\zeta_2^4 + 16\alpha^2\zeta_1^2 + 24\alpha^2\zeta_2^2 + 16\alpha^4)}{(10\alpha^2 + 2\zeta_1^2 + 3\zeta_2^2)(\alpha^2 + \zeta_1^2 + \zeta_2^2)(\alpha^2 + \zeta_1^2 + 2\zeta_2^2)}, \tag{67}$$

and Petrowsky well-posedness is evident. However, so is the loss of homogeneity if $\zeta_2 = 0$; see λ_3^2.

In the limit $\alpha \to 0$ we derive the following local conserved energy involving derivatives of fifth order:

$$\begin{aligned}
\overline{E}_0 = {} & 2\left\|\nabla\Delta\frac{\partial^2 u}{\partial x_2^2}\right\|^2 + 3\left\|\Delta\frac{\partial^3 u}{\partial x_2^3}\right\|^2 + \left\|\nabla\frac{\partial^4 u}{\partial x_2^4}\right\|^2 + 2\left\|\Delta\frac{\partial^3 u}{\partial x_2^2 \partial t}\right\|^2 \\
& + 2\left\|\nabla\frac{\partial^4 u}{\partial x_2^3 \partial t}\right\|^2 + 2\left\|\nabla\Delta\left(2\frac{\partial^2 u}{\partial t^2} - 2\Delta u - \frac{\partial^2 u}{\partial x_2^2}\right)\right\|^2 \\
& + 3\left\|\Delta\left(2\frac{\partial^3 u}{\partial x_2 \partial t^2} - 2\Delta\frac{\partial u}{\partial x_2} - \frac{\partial^3 u}{\partial x_2^3}\right)\right\|^2 + \left\|\nabla\left(2\frac{\partial^4 u}{\partial x_2^2 \partial t^2} - 2\Delta\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^4 u}{\partial x_2^4}\right)\right\|^2 \\
& + 2\left\|2\Delta\frac{\partial^3 u}{\partial t^3} + \frac{\partial^5 u}{\partial x_2^2 \partial t^3} - 2\Delta^2\frac{\partial u}{\partial t} - 2\Delta\frac{\partial^3 u}{\partial x_2^2 \partial t} - \frac{\partial^5 u}{\partial x_2^4 \partial t}\right\|^2.
\end{aligned} \tag{68}$$

Clearly this energy does not control all derivatives of fifth order. Now we consider three perturbations to (64).

In the first case we add $\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x_1}\right)^3$. The expressions for the λ_k^2 are now rather lengthy, but one can see that they can not be made positive for any choice of α. For example, in the special case $\zeta_2 = 0$:

$$\lambda_3^2 = \frac{(1+4\alpha)^5}{8\alpha\lambda_2^6}\frac{1}{q_0 + 32\alpha^4\zeta_1^4 + 72\alpha^3\zeta_1^4 - 76\alpha^2\zeta_1^4 + 28\alpha\zeta_1^4 - 16\zeta_1^6}, \tag{69}$$

$$q_0(\alpha,\zeta) = 160\alpha^8 + 360\alpha^7 + 324\alpha^6 + 146\alpha^5 + 33\alpha^4 + 3\alpha^3 + 192\alpha^6\zeta_1^2 + 368\alpha^5\zeta_1^2 + 112\alpha^4\zeta_1^2 - 44\alpha^3\zeta_1^2 - 16\alpha^2\zeta_1^2. \tag{70}$$

For ζ_1 large λ_3^2 is obviously not positive. Thus the Cauchy problem is ill-posed.

In the second case we add $\frac{\partial^3}{\partial x_1^2 \partial x_2}$. Then we find:

$$\lambda_3^2 = \frac{\alpha(10\alpha^2 + 2\zeta_1^2 + 3\zeta_2^2)^3}{2(q_1 + 64\alpha^4\zeta_1^4 + 164\alpha^4\zeta_2^4 + 4\alpha^2\zeta_1^2\zeta_2^4 + 6\alpha^2\zeta_2^6 - \zeta_1^4\zeta_2^2)}. \tag{71}$$

$$q_1(\alpha,\zeta) = 320\alpha^8 + 384\alpha^6\zeta_1^2 + 576\alpha^6\zeta_2^2 + 192\alpha^4\zeta_1^2\zeta_2^2. \tag{72}$$

Due to the sixth degree terms it is clear that for any fixed choice of α the denominator changes sign. Thus the Cauchy problem is ill-posed.

Finally[2], we add $\frac{\partial}{\partial t}$. Taking $\alpha = 1$, we find:

$$\lambda_1^2 = \frac{1}{4}, \quad \lambda_2^2 = \frac{16}{\tau_1}, \quad \lambda_3^2 = \frac{1}{4}\frac{\tau_1^2}{\tau_2}, \quad \lambda_4^2 = \frac{\tau_2}{\tau_1\tau_3},$$

$$\tau_1 = 19 + 4\zeta_1^2 + 6\zeta_2^2, \quad \tau_2 = 63 + 64{\zeta_1}^2 + 96{\zeta_2}^2 + 4{\zeta_2}^4,$$

$$\tau_3 = 2 + 2\zeta_1^2 + 3\zeta_2^2 + \zeta_1^4 + 3\zeta_1^2\zeta_2^2 + 2\zeta_2^4.$$

Well-posedness is evident. We also have that

$$\frac{d}{dt}\left(\widehat{E}_1 + \widehat{E}_2 + \widehat{E}_3 + \widehat{E}_4\right) = -2\tau_1\tau_2\tau_3|\hat{v}|^2, \tag{73}$$

where

$$\widehat{E}_1 = \frac{1}{4}\tau_1\tau_2\tau_3|\hat{v}|^2, \quad \widehat{E}_2 = \tau_2\tau_3\left|\frac{d\hat{v}}{dt} + 4\hat{v}\right|^2,$$

$$\widehat{E}_3 = \frac{1}{4}\tau_1\tau_3\left|4\frac{d\hat{v}}{dt}^2 + 16\frac{d\hat{v}}{dt} + \tau_1\hat{v}\right|^2,$$

$$\widehat{E}_4 = \left|\tau_1\frac{d^3\hat{v}}{dt^3} + 4\tau_1\frac{d^2\hat{v}}{dt^2} + \tau_4\frac{d\hat{v}}{dt} + \tau_2\hat{v}\right|^2,$$

and

$$\tau_4 = \left(106 + 54\zeta_1^2 + 81\zeta_2^2 + 4\zeta_1^4 + 12\zeta_1^2\zeta_2^2 + 10\zeta_2^4\right). \tag{74}$$

Localization of (73) is straightforward but the resulting expressions are lengthy.

6.2. *Equations of Mixed Type*

We now consider two systems of mixed type including parabolic and higher order dispersive terms. We note that symmetrizers for these and related problems have

[2]Peyser considers the perturbation $\frac{\partial}{\partial x_1}$. We have also used our technique to prove well-posedness in this case, but omit the lengthy algebraic expressions.

been explicitly constructed in Hagstrom and Lorenz (1995) and used to prove all-time existence results for nonlinear problems with small data.

6.2.1. The Linearized Navier–Stokes Equations. Our first example is a hyperbolic-parabolic system, the compressible Navier-Stokes equations linearized about a quiescent flow in two space dimensions. We note that the results can be easily generalized to three dimensions and to linearizations about a uniform flow via a Galilean transformation. See Hagstrom and Lorenz (1998, 2002) for restricted large data all-time existence theorems using the symmetrizer constructed in Hagstrom and Lorenz (1995).

Consider the equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0,\\
\frac{\partial u_1}{\partial t} + \frac{c^2}{\gamma}\left(\frac{\partial \rho}{\partial x} + \frac{\partial T}{\partial x}\right) &= \nu\left(\frac{4}{3}\frac{\partial^2 u_1}{\partial x^2} + \frac{1}{3}\frac{\partial^2 u_2}{\partial x \partial y} + \frac{\partial^2 u_1}{\partial y^2}\right),\\
\frac{\partial u_2}{\partial t} + \frac{c^2}{\gamma}\left(\frac{\partial \rho}{\partial y} + \frac{\partial T}{\partial y}\right) &= \nu\left(\frac{4}{3}\frac{\partial^2 u_2}{\partial y^2} + \frac{1}{3}\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial^2 u_2}{\partial x^2}\right),\\
\frac{\partial T}{\partial t} + (\gamma - 1)\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) &= \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),
\end{aligned}$$

with $\nu > 0$, $\kappa > 0$, and $\gamma > 1$. Introducing the 4-vector $w = (\rho \quad u_1 \quad u_2 \quad T)^T$ and performing a Fourier transformation in space we obtain the system:

$$\frac{d\hat{w}}{dt} = A\hat{w}, \tag{75}$$

where

$$A(\zeta) = -\begin{bmatrix} 0 & i\zeta_1 & i\zeta_2 & 0 \\ i\zeta_1 \frac{c^2}{\gamma} & \nu\left(\frac{4}{3}\zeta_1^2 + \zeta_2^2\right) & \frac{\nu}{3}\zeta_1\zeta_2 & i\zeta_1\frac{c^2}{\gamma} \\ i\zeta_2\frac{c^2}{\gamma} & \frac{\nu}{3}\zeta_1\zeta_2 & \nu\left(\frac{4}{3}\zeta_2^2 + \zeta_1^2\right) & i\zeta_2\frac{c^2}{\gamma} \\ 0 & i\zeta_1(\gamma-1) & i\zeta_2(\gamma-1) & \kappa\left(\zeta_1^2 + \zeta_2^2\right) \end{bmatrix}.$$

The minimal polynomial is $m_A(\lambda) = m_1(\lambda)m_3(\lambda)$ where

$$m_1(\lambda) = \lambda + \nu|\zeta|^2,$$

and introducing $\eta = 4\nu + 3\kappa$

$$m_3(\lambda) = \lambda^3 + \eta\frac{|\zeta|^2}{3}\lambda^2 + \frac{|\zeta|^2}{3}(4\kappa\nu|\zeta|^2 + 3c^2)\lambda + \frac{\kappa|\zeta|^4}{\gamma}.$$

Moreover we can show that:

$$m_3(\partial_t)\hat{v} = 0, \tag{76}$$

for $\hat{v} = \hat{\rho}, \widehat{T}, m_1(\partial_t)\hat{u}_i$. We will construct the energy based on the Sturm sequence for m_3. Further defining:

$$\tau(\zeta) = \beta + 4\gamma\kappa\nu\eta|\zeta|^2, \quad \beta = 3c^2(\gamma\eta - 3\kappa), \tag{77}$$

we find:

$$\lambda_1^2 = \frac{3}{\eta|\zeta|^2},$$
$$\lambda_2^2 = \frac{\gamma\eta^2}{\tau},$$
$$\lambda_3^2 = \frac{\tau}{3\kappa c^2\eta|\zeta|^2}.$$

Clearly, these coefficients have the correct sign under the assumptions made on the parameters. Clearing denominators and applying Parseval's equality we obtain the local energy equality:

$$\frac{dE}{dt} = -2\beta\left(\left\|\Delta\frac{\partial v}{\partial x_1}\right\|^2 + \left\|\Delta\frac{\partial v}{\partial x_2}\right\|^2\right) - 8\gamma\kappa\nu\eta\|\Delta^2 v\|^2, \tag{78}$$

$$\begin{aligned} E = {} & \frac{3\beta}{\eta}\|\Delta v\|^2 + 12\gamma\kappa\nu\left(\left\|\Delta\frac{\partial v}{\partial x_1}\right\|^2 + \left\|\Delta\frac{\partial v}{\partial x_2}\right\|^2\right) \\ & + \gamma\left(\left\|3\frac{\partial^2 v}{\partial x_1\partial t} - \eta\Delta\frac{\partial v}{\partial x_1}\right\|^2 + \left\|3\frac{\partial^2 v}{\partial x_2\partial t} - \eta\Delta\frac{\partial v}{\partial x_2}\right\|^2\right) \\ & \times \frac{1}{3c^2\eta\kappa}\left\|3\gamma\eta\frac{\partial^2 v}{\partial t^2} - \gamma\eta^2\Delta\frac{\partial v}{\partial t} - \beta\Delta v + 4\gamma\kappa\nu\eta\Delta^2 v\right\|^2. \end{aligned} \tag{79}$$

We can conclude from this equality the boundedness in time of all third order space derivatives of v, recalling that $v = \rho, T, \left(\frac{\partial u_i}{\partial t} - \nu\Delta u_i\right)$. It is interesting to note that one obtains the same estimates for ρ and T despite the fact that ρ is a hyperbolic variable while T is a parabolic variable in the language of Hagstrom and Lorenz (1995).

6.2.2. A Boussinesq System. Our final example is a Boussinesq system in $\mathbb{R}^2$. Consider the equations

$$\frac{\partial v}{\partial t} + \nabla(\eta + \delta\Delta\eta) = \nu\Delta v,$$
$$\frac{\partial \eta}{\partial t} + \nabla\cdot(v + \delta\Delta v) = 0,$$

with $\nu > 0$ and $\delta > 0$. Introducing $w = (v^T, \eta)^T$ and performing a Fourier transformation in space we obtain the system:

$$\frac{d\hat{w}}{dt} = A\hat{w}, \tag{80}$$

where

$$A(\zeta) = \begin{bmatrix} -v|\zeta|^2 & 0 & -i\zeta_1(1-\delta|\zeta|^2) \\ 0 & -v|\zeta|^2 & -i\zeta_2(1-\delta|\zeta|^2) \\ -i\zeta_1(1-\delta|\zeta|^2) & -i\zeta_2(1-\delta|\zeta|^2) & 0 \end{bmatrix}.$$

The minimal polynomial is $m_A(\lambda) = m_1(\lambda)m_2(\lambda)$ where

$$m_1(\lambda) = \lambda + v|\zeta|^2,$$

and

$$m_2(\lambda) = \lambda^2 + v|\zeta|^2\lambda + |\zeta|^2(\delta|\zeta|^2 - 1)^2.$$

Here $m_2(\partial_t)\eta = 0$ and $m_A(\partial_t)v_i = 0$, $i = 1, 2$. Forming the Sturm sequence of m_2 we obtain:

$$\lambda_1^2 = \frac{1}{v|\zeta|^2}, \quad \lambda_2^2 = \frac{v}{(\delta|\zeta|^2 - 1)^2}.$$

Clearing the denominators by multiplying by $|\zeta|^4(\delta|\zeta|^2 - 1)^2$ we obtain the following energy equality

$$\frac{d}{dt}\left(\left\|(1+\delta\Delta)^2\frac{\partial\eta}{\partial x_1}\right\|^2 + \left\|(1+\delta\Delta)^2\frac{\partial\eta}{\partial x_2}\right\|^2 + \left\|\frac{\partial\eta}{\partial t} - v\Delta\eta\right\|^2\right) \tag{81}$$

$$= -2v\|(1+\delta\Delta)\Delta\eta\|^2. \tag{82}$$

Using the PDE to replace η with v_1 and v_2 in the two first terms we obtain

$$\frac{d}{dt}\left(\left\|\frac{\partial v_1}{\partial t} - v\Delta v_1\right\|^2 + \left\|\frac{\partial v_2}{\partial t} - v\Delta v_2\right\|^2 + \left\|\frac{\partial\eta}{\partial t} - v\Delta\eta\right\|^2\right) \tag{83}$$

$$= -2v\left\|(1+\delta\Delta)\Delta\eta\right\|^2. \tag{84}$$

Thus we have a bound on the heat operator applied to all variables.

7. Conclusion

In conclusion, we have developed an automatic method for symmetrizing Cauchy problems which are at least weakly well-posed (or well-posed in the sense of Petrowsky) and applied it to derive energy equalities for a range of problems of physical interest. (See also Appelö et al., 2006.) This development leads to a number of questions we have not, as yet, considered. Prominent among these are:

i. Are the energies optimal in the sense that they yield the strongest possible estimates? For strictly hyperbolic problems we have shown this to be true, and it was also true in the other examples considered.

ii. Can the symmetrized system be used to analyze problems with variable coefficients and nonlinearities?

It is also of interest to use the energy equalities to develop admissible boundary conditions for mixed initial-boundary value problems.

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