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A STABLE FINITE DIFFERENCE METHOD FOR THE ELASTIC WAVE EQUATION ON COMPLEX DOMAINS WITH FREE SURFACE BOUNDARY CONDITIONS

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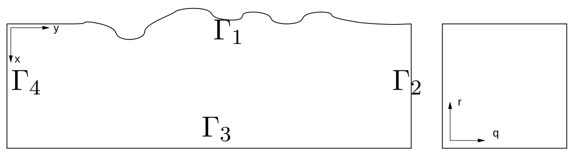

Figure 1: The Geometry. The surface is mapped onto to $q = 0$ and $y = 0$ is mapped onto $r = 0$.

Introduction

We consider elastic wave propagation on complex domains simulated by centered finite difference discretization of the elastic wave equation on second order form. Such discretizations are highly efficient but their use has been limited due to two major difficulties: Stable discretization of the free surface boundary condition and the handling of complex geometries needed for topography. A remedy to the first problem was recently presented in [1] were a stable discretization of the free surface boundary conditions on a Cartesian grid were derived. Here we generalize the results in [1] to curvilinear grids, providing a solution to the second difficulty.

In curvilinear coordinates equations the elastic wave equation is (on conservative form)

$$J\rho\ddot{u} = \Big[Jq_x\big[\,(2\mu+\lambda)\,\mathcal{D}^x u + \lambda\mathcal{D}^y v\big] + Jq_y\big[\mu(\mathcal{D}^x v + \mathcal{D}^y u)\big]\Big]_q \tag{1}$$

$$+\Big[Jr_x\big[\,(2\mu+\lambda)\,\mathcal{D}^x u + \lambda\mathcal{D}^y v\big] + Jr_y\big[\mu(\mathcal{D}^x v + \mathcal{D}^y u)\big]\Big]_r,$$

$$J\rho\ddot{v} = \Big[Jq_x\big[\mu(\mathcal{D}^x v + \mathcal{D}^y u)\big] + Jq_y\big[\,(2\mu+\lambda)\,\mathcal{D}^y v + \lambda\mathcal{D}^x u\big]\Big]_q \tag{2}$$

$$+\Big[Jr_y\big[(2\mu+\lambda)\mathcal{D}^y v + \lambda\mathcal{D}^x u\big] + Jr_x\big[\mu(\mathcal{D}^x v + \mathcal{D}^y u)\big]\Big]_r.$$

Here $\mathcal{D}^z w = (q_z w_q + r_z w_r)$, $z \in \{x, y\}$. We are interested in a domain (see Figure 1) with free surface boundary condition on Γ_1

$$q_x\left[(2\mu+\lambda)\mathcal{D}^x u + \lambda\mathcal{D}^y v\right] + q_y\mu(\mathcal{D}^x v + \mathcal{D}^y u) = 0, \tag{3}$$

$$q_y\left[(2\mu+\lambda)\mathcal{D}^x v + \lambda\mathcal{D}^y u\right] + q_x\mu(\mathcal{D}^x v + \mathcal{D}^y u) = 0, \tag{4}$$

periodic on Γ_2 and Γ_4 and homogeneous Dirichlet on Γ_3.

Spatial Discretization

Let the mapping from the unit square to the physical domain be given by

$$x(q,r), y(q,r), \ \ (q,r) \in [0,1]^2,$$

and let the grid in (q,r) space be defined by

$$q_i = (i-1)h_q, i = 0,\ldots,N_q, h_q = 1/(N_q - 1),$$
$$r_j = (j-1)h_r, j = 0,\ldots,N_r, h_r = 1/(N_r - 1).$$

Denote the grid functions $[u_{i,j}, v_{i,j}]$. The standard second order accurate difference operators are

$$h_r D_+^r u_{i,j} = u_{i,j+1} - u_{i,j}, \ \ D_-^r u_{i,j} = D_+^r u_{i,j-1},$$
$$h_q D_+^q u_{i,j} = u_{i+1,j} - u_{i,j}, \ \ D_-^q u_{i,j} = D_+^q u_{i-1,j},$$
$$2D_0^r u_{i,j} = D_+^r u_{i,j} + D_-^r u_{i,j}, \ 2D_0^q u_{i,j} = D_+^q u_{i,j} + D_-^q u_{i,j}.$$

We will also use the one sided operator and averaging operators

$$\widetilde{D_0^q} u_{i,j} = \begin{cases} D_+^q u_{i,j}, & i = 1, \\ D_0^q u_{i,j}, & i \geq 2, \end{cases}$$

$$2E_{1/2}^q(\sigma_{i,j}) = \sigma_{i+1,j} + \sigma_{i,j}, \ \ 2E_{1/2}^r(\sigma_{i,j}) = \sigma_{i,j+1} + \sigma_{i,j},$$

to discretize (1) and (2). The spatial approximation is

$$\begin{aligned} J\rho\ddot{u} &= D_-^q E_{1/2}^q(Jq_xq_x(2\mu+\lambda))D_+^q u + \widetilde{D_0^q}(Jq_xr_x(2\mu+\lambda))D_0^r u \\ &+ D_-^q E_{1/2}^q(Jq_xq_y\lambda)D_+^q v + \widetilde{D_0^q}(Jq_xr_y\lambda)D_0^r v + D_-^q E_{1/2}^q(Jq_yq_x\mu)D_+^q v \\ &+ \widetilde{D_0^q}(Jq_yr_x\mu)D_0^r v + D_-^q E_{1/2}^q(Jq_yq_y\mu)D_+^q u + \widetilde{D_0^q}(Jq_yr_y\mu)D_0^r u \\ &+ D_0^r(Jr_xq_x(2\mu+\lambda))\widetilde{D_0^q} u + D_-^r E_{1/2}^r(Jr_xr_x(2\mu+\lambda))D_+^r u \\ &+ D_0^r(Jr_xq_y\lambda)\widetilde{D_0^q} v + D_-^r E_{1/2}^r(Jr_xr_y\lambda)D_+^r v + D_0^r(Jr_yq_x\mu)\widetilde{D_0^q} v \\ &+ D_-^r E_{1/2}^r(Jr_yr_x\mu)D_+^r v + D_0^r(Jr_yq_y\mu)\widetilde{D_0^q} u + D_-^r E_{1/2}^r(Jr_yr_y\mu)D_+^r u \\ &\equiv L^{(u)}(u,v). \end{aligned} \tag{5}$$

$$\begin{aligned} J\rho\ddot{v} &= D_-^q E_{1/2}^q(Jq_xq_x\mu)D_+^q v + \widetilde{D_0^q}(Jq_xr_x\mu)D_0^r v + D_-^q E_{1/2}^q(Jq_xq_y\mu)D_+^q u \\ &+ \widetilde{D_0^q}(Jq_xr_y\mu)D_0^r u + D_-^q E_{1/2}^q(Jq_yq_x\lambda)D_+^q u + \widetilde{D_0^q}(Jq_yr_x\lambda)D_0^r u \\ &+ D_-^q E_{1/2}^q(Jq_yq_y(2\mu+\lambda))D_+^q v + \widetilde{D_0^q}(Jq_yr_y(2\mu+\lambda))D_0^r v + D_0^r(Jr_xq_x\mu)\widetilde{D_0^q} v \\ &+ D_-^r E_{1/2}^r(Jr_xr_x\mu)D_+^r v + D_0^r(Jr_xq_y\mu)\widetilde{D_0^q} u + D_-^r E_{1/2}^r(Jr_xr_y\mu)D_+^r u \\ &+ D_0^r(Jr_yq_x\lambda)\widetilde{D_0^q} u + D_-^r E_{1/2}^r(Jr_yr_x\lambda)D_+^r u \\ &+ D_0^r(Jr_yq_y(2\mu+\lambda))\widetilde{D_0^q} v + D_-^r E_{1/2}^r(Jr_yr_y(2\mu+\lambda))D_+^r v \\ &\equiv L^{(v)}(u,v). \end{aligned} \tag{6}$$

The discrete boundary conditions on Γ_3 and Γ_2, Γ_4 are

$$u_{N_q,j} = v_{N_q,j} = 0, \tag{7}$$

$$u_{i,1} = u_{i,N_r}, \;\; u_{i,0} = u_{i,N_r-1} \tag{8}$$

$$v_{i,1} = v_{i,N_r}, v_{i,0} = v_{i,N_r-1}. \tag{9}$$

Finally, in order to obtain a stable method we approximate the boundary conditions (3), (4) by

$$\begin{aligned}
&\frac{1}{2}\Big((Jq_xq_x(2\mu+\lambda))_{3/2,j}D_+^q u_{1,j}^1 + (Jq_xq_x(2\mu+\lambda))_{1/2,j}D_+^q u_{0,j}^1\Big) \\
&+(Jq_xr_x(2\mu+\lambda))_{1,j}D_0^r u_{1,j}^1 + \frac{1}{2}\Big((Jq_xq_y\lambda)_{3/2,j}D_+^q v_{1,j}^1 + (Jq_xq_y\lambda)_{1/2,j}D_+^q v_{0,j}^1\Big) \\
&+ (Jq_xr_y\lambda)_{1,j}D_0^r v_{1,j}^1 + \frac{1}{2}\Big((Jq_yq_x\mu)_{3/2,j}D_+^q v_{1,j}^1 + (Jq_yq_x\mu)_{1/2,j}D_+^q v_{0,j}^1\Big) \\
&+ (Jq_yr_x\mu)_{1,j}D_0^r v_{1,j}^1 + (Jq_yr_y\mu)_{1,j}D_0^r u_{1,j}^1 \\
&+ \frac{1}{2}\Big((Jq_yq_y\mu)_{3/2,j}D_+^q u_{1,j}^1 + (Jq_yq_y\mu)_{1/2,j}D_+^q u_{0,j}^1\Big) = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\frac{1}{2}\Big((Jq_xq_x\mu)_{3/2,j}D_+^q v_{1,j}^1 + (Jq_xq_x\mu)_{1/2,j}D_+^q v_{0,j}^1\Big) + \Big((Jq_xr_x\mu)_{1,j}D_0^r v_{1,j}^1\Big) \\
&+ \frac{1}{2}\Big((Jq_xq_y\mu)_{3/2,j}D_+^q u_{1,j}^1 + (Jq_xq_y\mu)_{1/2,j}D_+^q u_{0,j}^1\Big) + \Big((Jq_xr_y\mu)_{1,j}D_0^r u_{1,j}^1\Big) \\
&+ \frac{1}{2}\Big((Jq_yq_y(2\mu+\lambda))_{3/2,j}D_+^q v_{1,j}^1 + (Jq_yq_y(2\mu+\lambda))_{1/2,j}D_+^q v_{0,j}^1\Big) \\
&+ \Big((Jq_yr_y(2\mu+\lambda))_{1,j}D_0^r v_{1,j}^1\Big) + \Big((Jq_yr_x\lambda)_{1,j}D_0^r u_{1,j}^1\Big) \\
&+ \frac{1}{2}\Big((Jq_yq_x\lambda)_{3/2,j}D_+^q u_{1,j}^1 + (Jq_yq_x\lambda)_{1/2,j}D_+^q u_{0,j}^1\Big) = 0.
\end{aligned} \tag{11}$$

Introducing the discrete scalar products

$$(w,u)_h = \sum_{j=1}^{N_r-1} \frac{h_q h_r}{2} w_{1,j}u_{1,j} + \sum_{i=2}^{N_q-1} h_q h_r w_{i,j}u_{i,j},$$

$$(w,u)_h r = \sum_{j=1}^{N_r-1}\sum_{i=2}^{N_q-1} h_q h_r w_{i,j}u_{i,j},$$

where u and v are real valued functions, with corresponding norms $\|w\|_h^2 = (w,w)_h$, $\|w\|_{hr}^2 = (w,w)_{hr}$, we can prove the following

Lemma 1 (self adjoint spatial discretization). *For all real-valued grid functions* (u^0, v^0), (u^1, v^1) *satisfying the discrete boundary conditions (7), (8), (9), (10), (11), the spatial operator* $(L^{(u)}, L^{(v)})$ *is self-adjoint, i.e.*

$$\begin{aligned}
&(u^0, L^{(u)}(u^1,v^1))_h + (v^0, L^{(v)}(u^1,v^1))_h = \\
&(u^1, L^{(u)}(u^0,v^0))_h + (v^1, L^{(v)}(u^0,v^0))_h.
\end{aligned} \tag{12}$$

From lemma 1 it follows that.

Corollary 1 (conservation of energy). *All real-valued solutions* (u,v) *to the equations (5), (6) with boundary conditions (7), (8), (9), (10) and (11), satisfy*

$$\begin{aligned}
&\|\sqrt{J\rho}\,u_t\|_h^2 + \|\sqrt{J\rho}\,v_t\|_h^2 \\
&-(u, L^{(u)}(u,v))_h - (v, L^{(v)}(u,v))_h = C.
\end{aligned} \tag{13}$$

Here C is a constant depending only on the initial data.

We can also use the following lemma to establish that the conserved quantity (13) is a norm.

Lemma 2 (ellipticity). *For all real-valued grid functions* (u,v) *satisfying the discrete boundary conditions (7), (8), (9), (10), (11), the spatial operator the spatial operators* $L^{(u)}(u,v), L^{(v)}(u,v)$ *satisfy the following equality*

$$\begin{aligned}
-(u, L^{(u)}(u,v))_h - (v, L^{(v)}(u,v))_h &= \\
\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4&, \\
\mathcal{P}_1 \geq 0,\;\; \mathcal{P}_2 \geq 0,\;\; \mathcal{P}_3 \geq 0,\;\; \mathcal{P}_4 \geq 0&.
\end{aligned} \tag{14}$$

Where

$$\begin{aligned}
\mathcal{P}_1 &= \|\sqrt{J\lambda}(q_x\widetilde{D_0^q}u + r_xD_0^ru + q_y\widetilde{D_0^q}v + r_yD_0^rv)\|_h^2 + \|\sqrt{J2\mu}(q_x\widetilde{D_0^q}u + r_xD_0^ru)\|_h^2 \\
&+\|\sqrt{J2\mu}(q_y\widetilde{D_0^q}v + r_yD_0^rv)\|_h^2 + \|\sqrt{J\mu}(q_y\widetilde{D_0^q}u + r_yD_0^ru + q_x\widetilde{D_0^q}v + r_xD_0^rv)\|_h^2, \\
\mathcal{P}_2 &= \frac{h_r^2}{4}\Big(\|\sqrt{J2\mu}r_xD_+^rD_-^ru\|_h^2 + \|\sqrt{J2\mu}r_yD_+^rD_-^rv\|_h^2\Big) \\
&+\frac{h_q^2}{4}\Big(\|\sqrt{J2\mu}q_xD_+^qD_-^qu\|_{hr}^2 + \|\sqrt{J2\mu}q_yD_+^qD_-^qv\|_{hr}^2\Big) \\
&+\frac{h_r^2}{4}\|\sqrt{J\lambda}(r_xD_+^rD_-^ru + r_yD_+^rD_-^rv)\|_h^2 + \frac{h_q^2}{4}\|\sqrt{J\lambda}(q_xD_+^qD_-^qu + q_yD_+^qD_-^qv)\|_{hr}^2 \\
&\frac{h_r^2}{4}\|\sqrt{J\mu}(r_xD_+^rD_-^rv + r_yD_+^rD_-^ru)\|_h^2 + \frac{h_q^2}{4}\|\sqrt{J\mu}(q_xD_+^qD_-^qv + q_yD_+^qD_-^qu)\|_{hr}^2, \\
\mathcal{P}_3 &= \frac{h_r}{2}\sum_{j=1}^{N_r-1}\Big((J\lambda)_{N_q,j}((q_x)_{N_q,j}u_{N_q-1,j} + (q_y)_{N_q,j}v_{N_q-1,j})^2 \\
&+(J\mu)_{N_q,j}((q_x)_{N_q,j}v_{N_q-1,j} + (q_y)_{N_q,j}u_{N_q-1,j})^2\Big), \\
\mathcal{P}_4 &= \frac{h_qh_r}{2}\sum_{j=1}^{N_r-1}\Big((J\lambda)_{3/2,j}((q_x)_{3/2,j}D_+^qu_{1,j} + (q_y)_{3/2,j}D_+^qv_{1,j})^2 \\
&+(J\mu)_{3/2,j}((q_x)_{3/2,j}D_+^qv_{1,j} + (q_y)_{3/2,j}D_+^qu_{1,j})^2 \\
&+(J2\mu)_{3/2,j}(((q_x)_{3/2,j}D_+^qu_{1,j})^2 + ((q_y)_{3/2,j}D_+^qv_{1,j})^2)\Big).
\end{aligned}$$

Temporal Discretization

In time we discretize using standard Leap-Frog, i.e. the fully discrete equations are

$$\begin{aligned}
u^{n+1} - 2u^n + u^{n-1} &= (\rho J)^{-1}k^2L^{(u)}(u^n, v^n), \\
v^{n+1} - 2v^n + v^{n-1} &= (\rho J)^{-1}k^2L^{(v)}(u^n, v^n).
\end{aligned}$$

For this time discretization it can be proved that the discrete quantity

$$\begin{aligned}
C_e(t_{n+1}) = \|D_+^tu^n\|_\rho^2 + \|D_+^tv^n\|_\rho^2 - \\
(u^{n+1}, (\rho J)^{-1}L^{(u)}(u^n, v^n))_{\rho J} \\
-(v^{n+1}, (\rho J)^{-1}L^{(v)}(u^n, v^n))_{\rho J},
\end{aligned}$$

is conserved. That is, $C_e(t_{n+1}) = C_e(t_n)$. Here $(f, g/(\rho J))_{\rho J} = (f,g)_h$ is a weighted scalar product.

Computations

To verify the order of accuracy we determined forcing functions in the equations and boundary conditions such

N	maxerr u	maxerr v	e_i/e_{i+1}, u	e_i/e_{i+1}, v
80	0.16533	0.15609		
160	0.04245	0.03912	3.89	3.99
320	0.01071	0.00971	3.96	4.03
640	0.00269	0.00240	3.98	4.04

Table 1: Order of accuracy with method of manufactured solution.

that the solution is given by

$$u = \sin(6.2(x - 1.3t))\sin(6.2y),$$
$$v = \sin(6.2(x - 1.2t))\sin(6.2y).$$

We computed the solution on the grid defined by

$$x = x' + 0.05\sin(y'),\ y = y' + 0.05\sin(x'),$$
$$(x', y') \in [-\pi, \pi]^2.$$

We choose $\lambda = \mu = 1$ and advanced the solution up to time $\pi/5$ with a time step $k = 0.1h$, $h = \pi/N, N = 80, 160, 320, 640$ and computed the maximum error in the final solution. The results, showing the second order convergence, are displayed in Table 1.

Computations on a single grid

In Figure 2 we present a computation with initial data consisting of a pure pressure pulse centered in $(x, y) = (0.1, 1)$ which we advance up to time 0.2. The computation is performed for a material with $\rho = 1, \mu = 1, \lambda = 7$. As the pulse hits the free surface the P-wave transforms into reflected P and S-waves and surface waves traveling along the surface, see Figure 2. For this computation we monitored the quantity $C_e(t_{n+1}) - C_e(t_n)$ and as can be seen in Figure 3 it is conserved to machine precision.

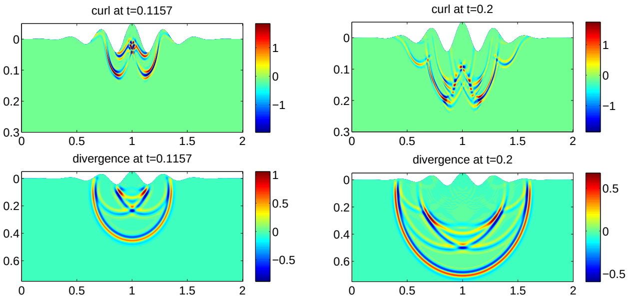

Figure 2: Divergence and Curl at two times. The initial data consisted of a pure pressure wave, thus there is no S-wave corresponding to the lowermost P-wave. The free surface boundary condition converts the P-wave into S-waves and surface waves.

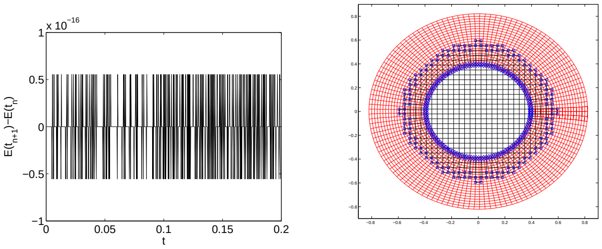

Figure 3: Left: Difference in the discrete energy, $C_e(t_{n+1}) - C_e(t_n)$, for subsequent time steps. Right: An overset grid.

Computations on an overset grid

Finally we present a computation on an overset grid of the type displayed in Figure 3. On the Cartesian grid we use the discretization from [1] and on the curvilinear grid we use the proposed discretization. To suppress weak instabilities triggered by the interpolation we add a small dissipative term, $\alpha h^3(D_+D_-)^2$, to both equations in both directions on both grids. Here we chose $\lambda = 0.005, \mu = 0.001$ and force the solution in $(x, y) = (-0.7, 0)$ with a Gaussian pressure pulse with support up to time 1.6. Again, the reflection of the free surface generates a S-wave and a P-wave while the primary P-wave traveling to the right in the interior remains solitary, see Figure 4.

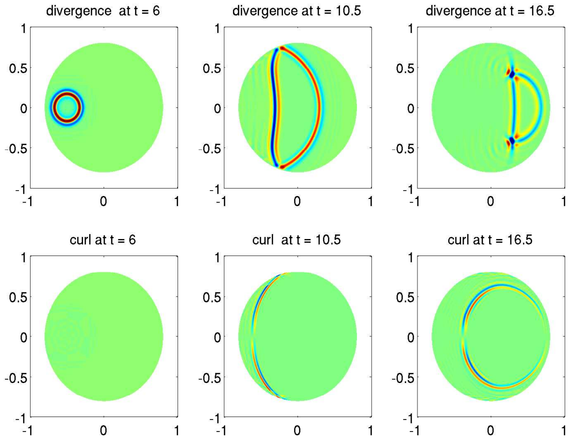

Figure 4: P and S-wave at three different times.

References

[1] Nilsson, Petersson, Sjogreen and Kreiss, “Stable difference approximations for the elastic wave equation in second order formulation”, submitted to SINUM.