Complex Variables

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ **(Cauchy − Riemann equations)**

Functions such as *u* and *v* which satisfy

$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ **(Laplace ' s equation)**

in *a* region *R* are called **harmonic functions.**

$\oint_C P\, dx + Q\, dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$ **(Green ' s Theorem in the Plane)**

$\oint_C B(z, \bar{z})\, dz = 2\, i \int\int_R \left(\frac{\partial B}{\partial \bar{z}} \right) dx\, dy$ **(Complex Form of Green ' s Theorem)**

Let $f(z)$ be analytic in *a* region $\mathcal{R}$ and on its boundary C. Then

$\oint_C f(z)\, dz = 0$ (**Cauchy ' s Theorem**).

Let $f(z)$ be continuous in *a* simply − connected (or or multiply − connected) region $\mathcal{R}$ and suppose that

$\oint_C f(z)\, dz = 0$

around every simple closed curve C in $\mathcal{R}$. Then $f(z)$ is analytic in $\mathcal{R}$ (**Morera ' s Theorem**).

Consequences of Cauchy ' s Theorem

Let $f(z)$ be analytic in *a* simply − closed region $\mathcal{R}$. Then the following theorems hold.

Theorem. If *a* and *z* are any two points in $\mathcal{R}$, then

$\int_a^z f(z)\, dz$

is *independent of the path* in $\mathcal{R}$ joining a and z.

Theorem. Let f (z) be analytic in a region bounded by the non − overlapping simple closed curves C, $C_1, C_2, C_3, \ldots, C_n$ [where $C_1, C_2, \ldots, C_n$ are inside C]. Then

$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \ldots + \oint_{C_n} f(z)\, dz.$