

Classical Mechanics

Lagrange 's Equations

Generalized Forces

$$dW = \sum_{\alpha=1}^n \Phi_{\alpha} dq_{\alpha},$$

$$\Phi_{\alpha} = \sum_{v=1}^N F_v \cdot \frac{\partial \mathbf{r}_v}{\partial q_{\alpha}} \text{ (called the **generalized force**)}$$

Lagrange 's Equations for Holonomic Systems

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_{\alpha}} \right) - \frac{\partial T}{\partial q_{\alpha}} = \Phi_{\alpha}$$

General Systems

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_{\alpha}} \right) - \frac{\partial L}{\partial q_{\alpha}} = \Phi_{\alpha} \text{ (here the generalized forces associated with the non – conservative forces in the system),}$$

$$L = T - V \text{ (called the **Lagrangian**)}$$

Generalized Momenta

$$p_{\alpha} = \frac{\partial T}{\partial \dot{q}_{\alpha}} = \frac{\partial L}{\partial \dot{q}_{\alpha}} \text{ (if the system is conservative)}$$

Lagrange 's Equations for Non – Holonomic Systems

Suppose we have $m < n$ equations of constraint that are non – integrable (*i.e.* **non – holonomic**) :

$$\sum_{\alpha} A_{\alpha} dq_{\alpha} + A dt = 0, \sum_{\alpha} B_{\alpha} dq_{\alpha} + B dt = 0, \dots \text{ (or an equivalent expression)}$$

Then we have for either non – holonomic or holonomic constraints :

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_{\alpha}} \right) - \frac{\partial T}{\partial q_{\alpha}} = \Phi_{\alpha} + \lambda_1 A_{\alpha} + \lambda_2 B_{\alpha} + \dots$$

If the forces are *conservative* then in terms of the *Lagrangian* we have :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_{\alpha}} \right) - \frac{\partial L}{\partial q_{\alpha}} = \lambda_1 A_{\alpha} + \lambda_2 B_{\alpha} + \dots$$

Hamiltonian Theory

The Principle of Virtual Work. A system of particles is in equilibrium iff

$$\sum_{v=1}^N F_v^{(a)} \cdot \delta \mathbf{r}_v = 0.$$

D' Alembert 's Principle. A system of particles moves in such a way that the total virtual work

$$\sum_{v=1}^N (F_v^{(a)} - \dot{p}_v) \cdot \delta \mathbf{r}_v = 0.$$

The Hamiltonian

$$H(p_{\alpha}, q_{\alpha}, t) = \sum_{\alpha=1}^n p_{\alpha} \dot{q}_{\alpha} - L$$

Hamilton's Equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

Calculus of Variations

For the integral

$$\int_a^b F(x, y, y') dx,$$

to be an *extremum*, it is a necessary condition that

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0 \text{ (called **Euler's equation**)}.$$

Central Forces

$F = f(r)\hat{r}$ (called a **central force**)

Properties

(i) $\mathbf{r} \cdot \mathbf{h} = 0$ (i.e. particle moves in a plane)

(ii) $\frac{dL}{dt} = 0$

(iii) $r^2 \dot{\theta} = 2 \dot{A}$ (**law of areas**)

Relating Orbits and Forces

$$\frac{d^2 u}{d\theta^2} + u = -\frac{1}{mh^2 u^2} f(1/u), \quad u = 1/r$$

Motion in an Inverse Square Field

If the law of central force is an inverse square law of attraction, then the path of the particle is a conic :

(i) $E < 0 \implies$ ellipse

(ii) $E = 0 \implies$ parabola

(iii) $E > 0 \implies$ hyperbola

Moving Coordinate Systems

Derivative Operators

$$D_F = D_M + \omega \times$$

Acceleration in a Moving System

$$D_F^2 \mathbf{r} = D_M^2 \mathbf{r} + \underbrace{(D_M \omega) \times \mathbf{r}}_{\text{linear acceleration}} + \underbrace{2 \omega \times D_M \mathbf{r}}_{\text{Coriolis acceleration}} + \underbrace{\omega \times (\omega \times \mathbf{r})}_{\text{centripetal acceleration (= -centrifugal force)}}$$

Space Motion of Rigid Bodies

$$I = \sum m r^2 \text{ or } I = \int r^2 dm$$

$$K^2 = \frac{I}{M} \text{ (radius of gyration)}$$

$I = I_c + Mb^2$ (**Parallel Axis Theorem**)

$I_z = I_x + I_y$ (**Perpendicular Axes Theorem**)

Kinetic Energy of Rotation

$$T = \frac{1}{2} (\omega \cdot L) = \frac{1}{2} (I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + I_{zz} \omega_z^2 + 2 I_{xy} \omega_x \omega_y + 2 I_{xz} \omega_x \omega_z + 2 I_{yz} \omega_y \omega_z)$$

Rigid Body with One Point Fixed

$$v_v = \omega \times r_v$$

$$L = \sum m_v (r_v \times v_v)$$

The Inertia Tensor

Moment of Inertia Tensor

$$I \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Angular Momentum

$$L = I\omega$$

Moments and Products of Inertia

$$I_{xx} = m_v (y_v^2 + z_v^2)$$

$$I_{yy} = m_v (x_v^2 + z_v^2)$$

$$I_{zz} = m_v (x_v^2 + y_v^2)$$

$$I_{xy} = I_{yx} = -m_v x_v y_v$$

$$I_{yz} = I_{zy} = -m_v y_v z_v$$

$$I_{zx} = I_{xz} = -m_v x_v z_v$$

Principal Axes of Inertia

Principal Moments of Inertia

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$$

Angular Momentum

$$L_i = I_i \omega_i$$

Kinetic Energy

$$T = \frac{1}{2} (\omega \cdot L) = \frac{1}{2} (I_i \omega_i^2)$$

Ellipsoid of Inertia

$$I = I_n = m_i (r_i \times n)^2 = I_{xx} \cos^2 \alpha + I_{yy} \cos^2 \beta + I_{zz} \cos^2 \gamma + 2 I_{xy} \cos \alpha \cos \beta + 2 I_{xz} \cos \alpha \cos \gamma + 2 I_{yz} \cos \beta \cos \gamma,$$

$$n = (\cos \alpha, \cos \beta, \cos \gamma)$$

$$I_{xx} \rho_x^2 + I_{yy} \rho_y^2 + I_{zz} \rho_z^2 + 2 I_{xy} \rho_x \rho_y + 2 I_{yz} \rho_y \rho_z + 2 I_{zx} \rho_z \rho_x = 1,$$

$$\rho = n / \sqrt{I_n}$$

The Euler Angles

Angular Velocity

$$(\text{using } \omega = \omega_\phi k + \omega_\theta I' + \omega_\psi K')$$

$$\omega_{x'} = \omega_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \sin \psi$$

$$\omega_{y'} = \omega_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$$

$$\omega_{z'} = \omega_3 = \dot{\phi} \cos \theta + \dot{\psi}$$

Equations of Motion

$$I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = \tau_1$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 = \tau_2$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = \tau_3$$

Small Oscillations about Positions of Equilibrium

Approximate Expressions for T , P , V

$$T = \frac{1}{2} a_{r1} \dot{q}_r \dot{q}_l,$$

$$a_{r1} = m_1 \left(\frac{\partial x_1}{\partial q_r} \frac{\partial x_1}{\partial q_l} + \frac{\partial y_1}{\partial q_r} \frac{\partial y_1}{\partial q_l} \right) + m_2 \left(\frac{\partial x_2}{\partial q_r} \frac{\partial x_2}{\partial q_l} + \frac{\partial y_2}{\partial q_r} \frac{\partial y_2}{\partial q_l} \right)$$

$$P = -\frac{1}{2} b_{r1} \dot{q}_r \dot{q}_l,$$

$$b_{r1} = a_{r1} (m_1 = b_1, m_2 = b_2)$$

$$V = \frac{1}{2} c_{r1} q_r q_l,$$

$$c_{r1} = \left(\frac{\partial^2 V}{\partial q_r \partial q_l} \right)_0$$

Differential Equations of Motion

$$a_{11} \ddot{q}_1 + \underbrace{b_{11} \dot{q}_1}_{\text{viscous forces}} + \underbrace{c_{11} q_1}_{\text{conservative forces}} + a_{12} \ddot{q}_2 + b_{12} \dot{q}_2 + c_{12} q_2 + a_{13} \ddot{q}_3 + b_{13} \dot{q}_3 + c_{13} q_3 = 0$$

$$a_{21} \ddot{q}_1 + b_{21} \dot{q}_1 + c_{21} q_1 + a_{22} \ddot{q}_2 + b_{22} \dot{q}_2 + c_{22} q_2 + a_{23} \ddot{q}_3 + b_{23} \dot{q}_3 + c_{23} q_3 = 0$$

$$a_{31} \ddot{q}_1 + b_{31} \dot{q}_1 + c_{31} q_1 + a_{32} \ddot{q}_2 + b_{32} \dot{q}_2 + c_{32} q_2 + a_{33} \ddot{q}_3 + b_{33} \dot{q}_3 + c_{33} q_3 = 0$$

Solutions of the Equations of Motion : Conservative Forces Only

$$q_1 = A \cos(\omega t + \phi), \quad q_2 = B \cos(\omega t + \phi), \quad q_3 = C \cos(\omega t + \phi),$$

$$A = c_1 d_{11}, \quad B_1 = c_1 d_{21}, \quad C_1 = c_1 d_{31},$$

$$d_{rc} = (-1)^{r+c} M_{rc} \text{ (called the cofactor)}$$

$$D = \begin{vmatrix} a_{11} \omega^2 - c_{11} & a_{12} \omega^2 - c_{12} & a_{13} \omega^2 - c_{13} \\ a_{21} \omega^2 - c_{21} & a_{22} \omega^2 - c_{22} & a_{23} \omega^2 - c_{23} \\ a_{31} \omega^2 - c_{31} & a_{32} \omega^2 - c_{32} & a_{33} \omega^2 - c_{33} \end{vmatrix} = 0$$