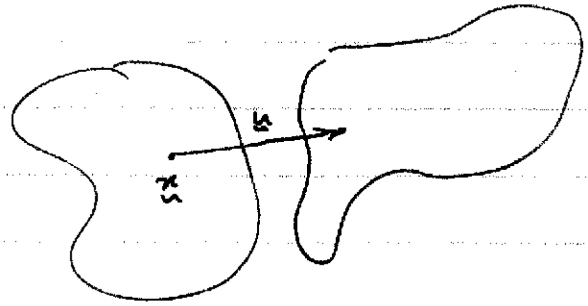
## 2. Elastodynamics

Kinematics



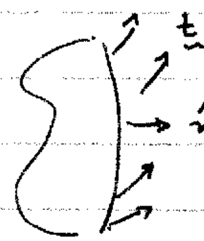

Displacement $\underset{\sim}{u}(x,t)$

Strain $\underset{\sim}{\varepsilon} = \frac{1}{2}(\nabla u + \nabla u^T)$

Particle velocity $\underset{\sim}{v} = \frac{\partial u}{\partial t}$

Balance Laws.

1. Contact force $\underset{\sim}{t}(\hat{n}, x, t)$
2. Body force $\underset{\sim}{f}(\underset{\sim}{x}, t)$




Euler's postulate. For any part of the body $P \subset \Omega$

$$\frac{d}{dt}\int_P \rho \dot{\underset{\sim}{u}}\, dV = \int_{\partial P} \underset{\sim}{t}\, da + \int_P \underset{\sim}{f}\, dV$$

$$\frac{d}{dt}\int_P \underset{\sim}{x} \times \rho \dot{\underset{\sim}{u}}\, dV = \int_{\partial P} \underset{\sim}{x} \times \underset{\sim}{t}\, da + \int_P \underset{\sim}{x} \times \underset{\sim}{f}\, dV$$

$$\Rightarrow \quad \underset{\sim}{t}(\hat{n}, \underset{\sim}{x}, t) = \underset{\sim}{T}(\underset{\sim}{x}, t)\, \hat{n}$$

$\underset{\sim}{T} = \underset{\sim}{T}^T$ ↖ Cauchy stress

$\rho \ddot{\underset{\sim}{u}} = \operatorname{div} \underset{\sim}{T} + \underset{\sim}{f}$

Constitutive Relation $\underset{\sim}{T} = \mathbb{C}\underset{\sim}{\varepsilon}$

isotropic: $T_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$

$\Rightarrow \quad \rho \ddot{u} = \operatorname{div} \mathbb{C} \nabla u \qquad$ if $\underset{\sim}{f} = 0$

or $\boxed{\begin{aligned} \rho \ddot{u} &= \mu \Delta u + \frac{\mu}{1-2\nu} \nabla(\operatorname{div} u) \\ \rho \ddot{u}_i &= \mu u_{i,jj} + \frac{\mu}{1-2\nu} u_{j,ji} \end{aligned}}$ if isotropic

Cauchy - Navier Equations.

<u>Plane waves</u> in an infinite homogeneous isotropic elastic medium.

A plane wave is a displacement field of the form

$$\underset{\sim}{u}(x,t) = \underset{\sim}{a}\, f(x \cdot \hat{m} - ct)$$

$\underset{\sim}{a}$ constant vector

$\hat{m}$ unit vector

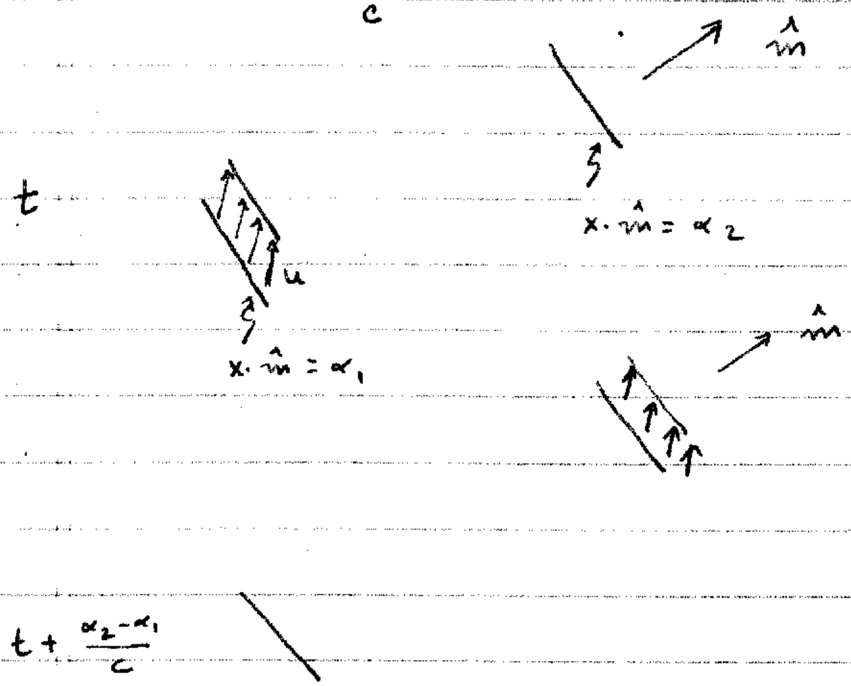
$f$ scalar function of one variable

$$u_i(\underset{\sim}{x},t) = a_i\, f(x \cdot \hat{m} - ct)$$

Note 1. The displacement is always parallel to $\underline{a}$

2. For any given time $t$ and any $\alpha$, the displacement is constant on the plane $\{x \cdot \hat{m} = \alpha\}$

3. ~~The displacement~~ For any given $\alpha_2 > \alpha_1$, the displacement of the plane $\{x \cdot \hat{m} = \alpha_1\}$ at time $t$ is exactly the same as the displacement of the plane $\{x \cdot \hat{m} = \alpha_2\}$ at time $t + \frac{\alpha_2 - \alpha_1}{c}$




So, the wave propagates in the direction $\hat{m}$ at ~~the~~ speed $c$.

$\nabla u$ : $\quad u_{i,j} = a_i m_j f'(x\cdot\hat{m} - ct)$

div $u$ = tr $\varepsilon$ : $\quad u_{i,i} = a_i m_i f'(x\cdot\hat{m} - ct)$

Curl $u$ : $\quad \varepsilon_{ijk} u_{j,k} = \varepsilon_{ijk} a_j m_k f'(x\cdot\hat{m} - ct)$

$\Delta u$ : $\quad u_{i,jj} = a_i f''(x\cdot\hat{m} - ct)$

$\nabla(\text{div } u)$ : $\quad u_{j,ji} = a_j m_j m_i f''(x\cdot\hat{m} - ct)$

$\dot{u}$ : $\quad \dot{u}_i = -c\, a_i f'(x\cdot\hat{m} - ct)$

$\ddot{u}$ : $\quad \ddot{u}_i = c^2 a_i f''(x\cdot\hat{m} - ct)$

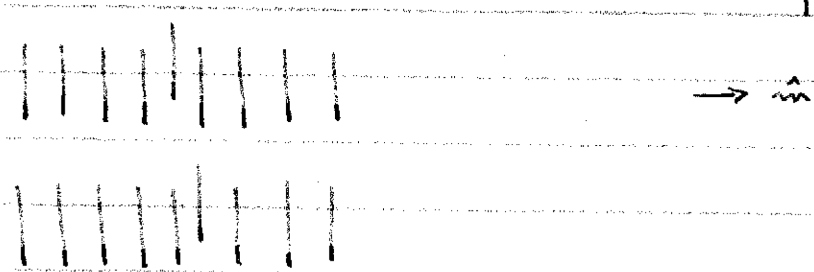
$\therefore$ Cauchy - Navier Equation

$$\rho c^2 a_i f'' = \mu a_i f'' + \frac{\mu}{1-2\nu} a_j m_j m_i f''$$

Assuming that $f'' \not\equiv 0$

$$(\mu - \rho c^2)\underset{\sim}{a} + \frac{\mu}{1-2\nu}(\underset{\sim}{a}\cdot\hat{m})\hat{m} = 0$$

$\Rightarrow$ Either (i) $(a\cdot\hat{m}) = 0$, $\mu = \rho c^2$ or $\boxed{c_s^2 = \frac{\mu}{\rho}}$




$\therefore$ purely shear deformation

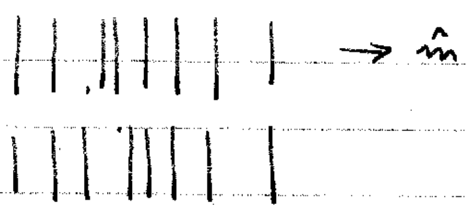
vol. strain = tr $\underset{\sim}{\varepsilon}$ = $a\cdot\hat{m} f'' = 0$.

These are called shear, rotational, transverse, secondary or S- waves.

or (ii) $\underline{a \parallel \hat{m}}$ or $a = (a \cdot \hat{m})\,\hat{m}$

$$\therefore \quad \mu - \rho c^2 + \frac{\mu}{1-2\nu} = 0$$

$$\text{or} \quad \boxed{c_p^2 = \frac{2(1-\nu)}{1-2\nu}\,\frac{\mu}{\rho} = \frac{\lambda+2\mu}{\rho} = \frac{3\kappa}{\rho}}$$



$$\text{curl } u = \underset{\sim}{a} \times \hat{m}\, f'' = 0$$

.... irrotational deformation.

These are called pressure, volumetric, longitudinal primary or p- wave.

Note: $\dfrac{c_p^2}{c_s^2} = \dfrac{2(1-\nu)}{1-2\nu} \geq 1$ for $\nu \in (0, \frac{1}{2})$.

$\therefore$ P-waves travel faster than S-waves.

Summary: One can have only <u>two</u> types of plane waves in isotropic elastic media:
Longitudinal P-waves travelling at $c_p$ or
Transverse S-waves travelling at $c_s$.
No mixed waves.

## Some comments concerning the Cauchy-Navier Equations of elastodynamics

Let $\underset{\sim}{w} = \text{curl}\, \underset{\sim}{u}$, $\theta = \text{tr}\, \varepsilon = \text{div}\, u$

(i) $w = 0 \Rightarrow \underset{\sim}{u}_{tt} = c_p^2 \Delta \underset{\sim}{u}$

(ii) $\theta = 0 \Rightarrow \underset{\sim}{u}_{tt} = c_s^2 \Delta \underset{\sim}{u}$

(iii) $\theta_{tt} = c_p^2 \Delta \theta$

(iv) $\underset{\sim}{w}_{tt} = c_s^2 \Delta \underset{\sim}{w}$

Green-Lamé Solution. $\underset{\sim}{u}$ is a solution of the Cauchy-Navier equation if and only if

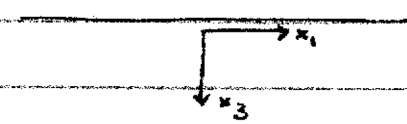
$$\underset{\sim}{u} = \nabla\varphi + \text{curl}\, \underset{\sim}{v}$$

where $\varphi$, $\underset{\sim}{v}$ satisfy

$$\varphi_{tt} = c_p^2 \Delta\varphi, \quad \underset{\sim}{v}_{tt} = c_s^2 \Delta \underset{\sim}{v}, \quad \text{div}\, \underset{\sim}{v} = 0.$$

## Rayleigh Surface Waves

Consider a half space $\{x_3 > 0\}$ with a traction free surface.




We seek a wave-like solution of the Cauchy Navier equation that propagates in the $x_1$-direction and whose amplitude decays in the $x_3$-direction.

Thus we seek a solution to the Cauchy - Navier equation such that

1. $u_2 = 0$, $u_{2,i} = 0$.
2. $u_i \to 0$ as $x_3 \to \infty$
3. $\underset{\sim}{t} = 0$ on $\{x_3 = 0\}$ i.e., $T_{3i} = 0$ on $\{x_3 = 0\}$

Recalling the Green - Lamé solution and in accordance to (1): $\varphi = \varphi(x_1, x_3, t)$

$$\underset{\sim}{\psi} = \begin{pmatrix} 0 \\ \psi(x_1, x_3, t) \\ 0 \end{pmatrix}$$

where $\varphi_{tt} = c_p^2 \left( \varphi_{,11} + \varphi_{,33} \right)$

$$\psi_{tt} = c_s^2 \left( \psi_{,11} + \psi_{,33} \right)$$

Rayleigh ansatz: $\varphi = f(x_3) \exp\left(ik(x_1 - ct)\right)$ $\qquad$ $c, k$ arbitrary constants

$$\psi = g(x_3) \exp\left(ik(x_1 - ct)\right)$$

$$\Rightarrow \quad f'' - k^2\left(1 - \frac{c^2}{c_p^2}\right) f \quad , \quad g'' - k^2\left(1 - \frac{c^2}{c_s^2}\right) g$$

(2) $\Rightarrow$ $0 < c < c_s$ since $c_s < c_p$

$\Rightarrow$

$$f = A \exp(-\kappa_1 x_3) \quad , \quad g = B \exp(-\kappa_2 x_3)$$

$A, B$ arbitrary complex constants

$$\kappa_1 = k\sqrt{1 - \frac{c^2}{c_p^2}} > 0 \, , \qquad \kappa_2 = k\sqrt{1 - \frac{c^2}{c_s^2}} > 0$$

Recalling $\underset{\sim}{u} = \nabla\varphi + \operatorname{curl} v = \begin{pmatrix} \varphi_{,1} - \psi_{,3} \\ 0 \\ \varphi_{,3} + \psi_{,1} \end{pmatrix}$

$$u_1 = \left(ik\, A \exp(-\kappa_1 x_3) + \kappa_2 B \exp(-\kappa_2 x_3)\right) \exp\left(ik(x_1 - ct)\right)$$

$$u_3 = \left(-\kappa_1 A \exp(-\kappa_1 x_3) + ik\, B \exp(-\kappa_2 x_3)\right) \exp\left(ik(x_1 - ct)\right)$$

We seek to find $A, B, c, k$ from (3).

Recall,

$$T_{31} = \mu\left(u_{3,1} + u_{1,3}\right), \quad T_{32} = 0,$$

$$T_{33} = \lambda\left(u_{1,1} + u_{3,3}\right) + 2\mu\, u_{3,3}$$

$$(3) \Rightarrow \begin{cases} 2ik\kappa_1 A + \left(2 - \dfrac{c^2}{c_s^2}\right) k^2 B = 0 \\ \left(2 - \dfrac{c^2}{c_s^2}\right) ik A + 2\kappa_2 B = 0 \end{cases}$$

Note that these relations are independent of $k$.

To have a non-trivial solution,

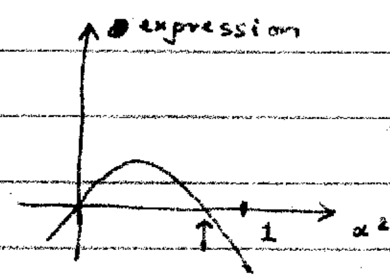
$$\left(2 - \frac{c^2}{c_s^2}\right)^2 - 4\sqrt{\left(1 - \frac{c^2}{c_p^2}\right)}\sqrt{\left(1 - \frac{c^2}{c_s^2}\right)} = 0$$

Set $\alpha = c/c_s$, $\beta = \dfrac{c_s}{c_p}$

$$(2 - \alpha^2)^2 - 4\sqrt{(1 - \beta^2\alpha^2)(1 - \alpha^2)} = 0$$

Exactly one root $\alpha \in (0, 1)$.

Depends on $\beta$ or on $\nu$

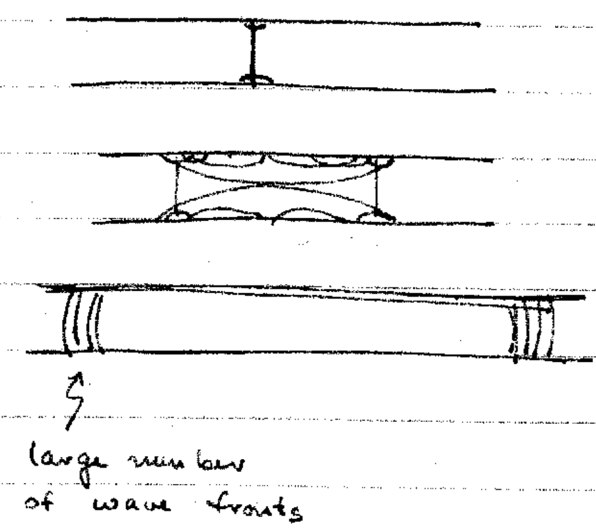



Can now find $A/B$

## 3. Wave guide

Consider a bar as a three-dimensional elastic solid. Note that the bar wave speed, $c_0 = \sqrt{\frac{E}{\rho}}$ is distinct from the longitudinal wave speed $c_p$. Infact one can verify that longitudinal plane waves do <u>not</u> satisfy the stress-free lateral boundary conditions of a bar. Conversely, one can show that bar waves do not solve the three-dimensional equations of elastodynamics.

Bar waves arise because bars act as "wave guides". An initial pulse is reflected numerous times from the boundary and at long times one only sees a cluster of wave fronts. Each wave front propagates according to 3-d elastodynamics, but the cluster propagates differently.



Dispersive waves in one (space) dimension

Suppose we have a linear equation

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\varphi = 0$$

where $P(\bar{X}, T)$ is some polynomial

Eg: 1. $P(\bar{X},T) = \bar{X}^2 - \frac{1}{c^2}T^2$ in simple wave eqn.

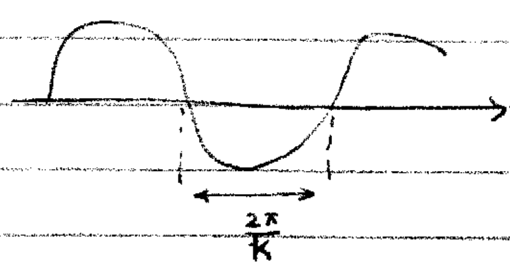
2. $P(\bar{X},T) = EI\bar{X}^4 + \rho A T^2$ ~~in~~ ~~Eq~~ for the deflection ~~lateral motion~~ in a small deflection Euler-Bernoulli beam.

3. $P(\bar{X},T) = T^2 - c_0^2\bar{X}^2 - \frac{1}{2}\nu^2 a^2 \bar{X}^2 T^2$ for a solid bar with round cross-section.

Try: $\varphi(x,t) = A\exp\left(i(kx - \omega t)\right)$

↑ frequency (rad/time)

↑ wave number (rad/length)




Substituting in the equation, $P(ik, -i\omega) = 0$

Solve: $\omega = \omega_l(k)$ $l = 1, \ldots, n$

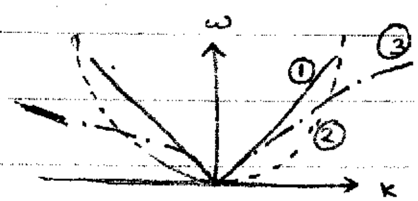
"Dispersion relation"

depends on order of polynomial

Eg: 1. $c^2 k^2 = \omega^2$ or $\omega = \pm c k$

2. $\gamma^2 k^4 = \omega^2$ or $\omega = \pm \gamma k^2$

$\gamma^2 = \dfrac{EI}{\rho A}$



3. $\omega^2 = c_0^2 k^2 - \frac{1}{2} a^2 \rho^2 \omega^2 k^2$ or $\omega = \pm \dfrac{k}{\sqrt{1 + \frac{1}{2} a^2 \rho^2 k^2}}$

(Note that bar equation is good for waves with small wave number; i.e., long wave lengths)

Terminology:

Phase: $\theta(x, t) = kx - \omega t$

Instantaneous wave number $\dfrac{\partial \theta}{\partial x}$ Inst. frequency $-\dfrac{\partial \theta}{\partial t}$

Points of constant phase propagate with velocity: $\theta(x(t), t) = \theta_0$

$\therefore \theta_x \dot{x} + \theta_t = 0 \Rightarrow \dot{x} = \dfrac{-\theta_t}{\theta_x} = \dfrac{\omega}{k}$

$$\Rightarrow \boxed{c_{phase} = \frac{\omega}{k} \qquad \text{"phase velocity"}}$$

Velocity of points of constant wave number: $k(x(t), t) = k_0$

$k_x \dot{x} + k_t = 0$

$k_t = \theta_{xt} = -\omega_x = \dfrac{d\omega}{dk} k_x$

$\therefore k_x \dot{x} - \dfrac{d\omega}{dk} k_x = 0 \Rightarrow \dot{x} = \dfrac{d\omega}{dk}$

$$\Rightarrow \boxed{c_{group} = \frac{d\omega}{dk} \quad \text{"group velocity"}}$$

It can be shown that

- wave packets
- energy

propagate with the group velocity.

## Wave packets

$$u(x,t) = \int_{-\infty}^{\infty} g(k) \exp\left(i(kx - \omega(k)t)\right) dk$$

Suppose the waves are clustered near a wave number $k_0$.

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}(k_0)(k-k_0) + \frac{1}{2}\frac{d^2\omega}{dk^2}(k_0)(k-k_0)^2 + \cdots$$

If $\frac{d^2\omega}{dk^2}(k_0)(k-k_0)^2 t \ll 1$, then

$$u(x,t) \simeq \int_{-\infty}^{\infty} g(k) \exp\left(i(k_0x - \omega(k_0)t) + i\left((k-k_0) - \frac{d\omega}{dk_0}(k_0)(k-k_0)\right)\right) dk$$

$$= \underbrace{\exp\left(i(k_0x - \omega(k_0))\right)}\ \underbrace{\int_{-\infty}^{\infty} g(k) \exp\left(i\left((k-k_0) - \frac{d\omega}{dk}(k_0)(k-k_0)\right)\right) dk}$$

wave with velocity $\frac{\omega(k_0)}{k_0}$

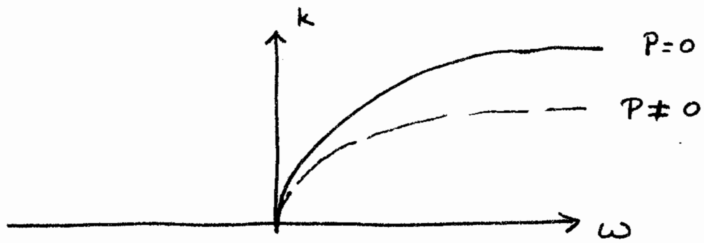
modulated by a wave with velocity $\frac{d\omega}{dk}(k_0)$

and ~~wave number $k_0$~~ $k$

$\exp\left(i(kx-\omega t)\right)$

$$\Rightarrow \rho I k^2 \omega^2 = EI k^4 + k^2 P - \rho A \omega^2$$

$$\omega^2 \left(\rho I k^2 + \rho A\right) = EI k^4 + P k^2 \quad \text{or} \quad \omega^2 = \frac{EI k^4 + P k^2}{\rho A + \rho I k^2}$$

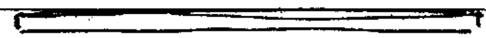



P<0 ... not well posed for small k

# Stress Waves

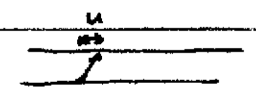


## 1. Waves in bars.



Bars are rods capable of sustaining only uniaxial stress and ~~[illegible]~~ longitudinal motion.

Kinematics: Displacement $u(x,t)$



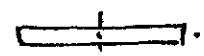

Strain $\varepsilon = \dfrac{\partial u}{\partial x} = u' = u_x$

Particle velocity $v = \dfrac{\partial u}{\partial t} = \dot{u} = u_t$

Compatibility: $v_x = \varepsilon_t$

Balance of linear momentum

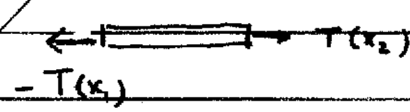
~~[illegible]~~ Forces : 1. Contact force $T(x)$



2. Body force $f(x)$ per unit length


Euler's postulate: The rate of change of the momentum of any portion of a bar is equal to the external forces acting on it. Consider a portion $(x_1, x_2)$:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho A \dot{u}\, dx = T(x_2) - T(x_1) + \int_{x_1}^{x_2} f\, dx$$




density

cross-sectional area

$$\therefore \int_{x_1}^{x_2} \rho A \ddot{u}\, dx = \int_{x_1}^{x_2} \left( \frac{\partial T}{\partial x} + f \right) dx$$

$$\Rightarrow \boxed{\rho A \ddot{u} = \frac{\partial T}{\partial x} + f}$$

Constitutive relation: $T = EA\varepsilon = EA\,u'' = EA\,u_{xx}$

$\therefore$ Equation of a linear elastic bar with no body force:

$$\rho \ddot{u} = E u'' \quad \text{or} \quad \rho u_{tt} = E u_{xx}$$

$$\text{or} \quad \boxed{u_{tt} = c^2 u_{xx}} \qquad c^2 = \frac{E}{\rho}$$

.... wave equation.

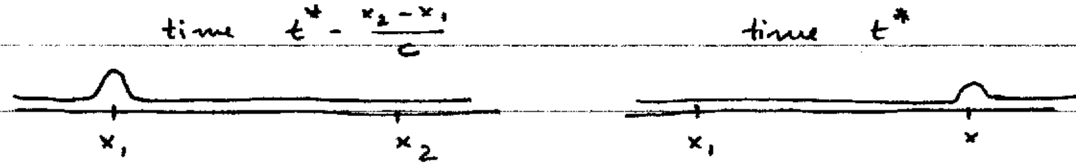
The solution to this equation can be written as:

$$u(x,t) = f(x-ct) + g(x+ct)$$

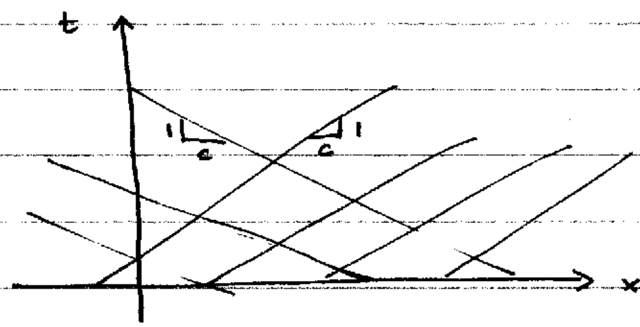
where $f, g$ ~~is~~ are smooth (twice cont. diff. functions.

.... superposition of a forward moving wave of velocity $c$ and a backward moving wave of velocity $c$.

Let $x_1 < x_2$. If $u(x,t) = f(x-ct)$, then the displacement at $x_2$ at time $t^*$ is exactly the same as the displacement at $x_1$ at time $t^* - \frac{x_2 - x_1}{c}$. Therefore, the displacement moves forward in an undistorted <u>wave</u> at speed $c$




$\therefore$ $f(x-ct)$ is a forward moving wave
$g(x+ct)$ is a backward moving wave.

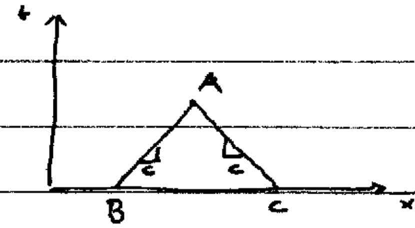



Note: Displacement (information) is propagated forward or backward at speed $c$; or it is propagated along lines

$$x - ct = \text{const}$$
$$x + ct = \text{const}$$

.... "characteristics".

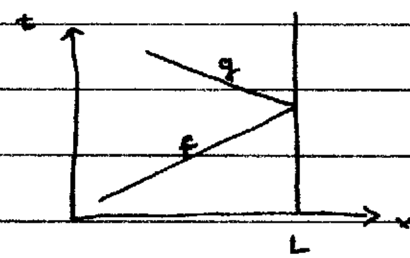
Note: The displacement at A can only be affected by "events" in ABC. It is completely oblivious to events outside.




Boundary conditions

a/ rigid boundary

$u(L,t) = 0$




$\therefore\ f(L-ct) + g(L+ct) = 0$

or $f(L-ct) = -g(L+ct) \quad \forall t.$

$\therefore$ a forward moving wave is bounced off with opposite displacement.

However,

$+\cancel{c}\, f'(L-ct) = +\cancel{c}\, g'(L+ct)$

$\therefore$ the strain of the forward moving wave has the same sign as that of the backward moving wave:

tensile in $\Rightarrow$ tensile back

comp. in $\Rightarrow$ comp. back

b/ free boundary : $T(L,t) = 0 \Rightarrow \varepsilon(L,t) = 0$

$\therefore \quad f'(L-ct) + g'(L+ct) = 0$

$\therefore \quad f'(L-ct) = -g'(L+ct)$

or a forward moving wave is bounced off with strain of the opposite sign.

tensile in $\Rightarrow$ compressive back

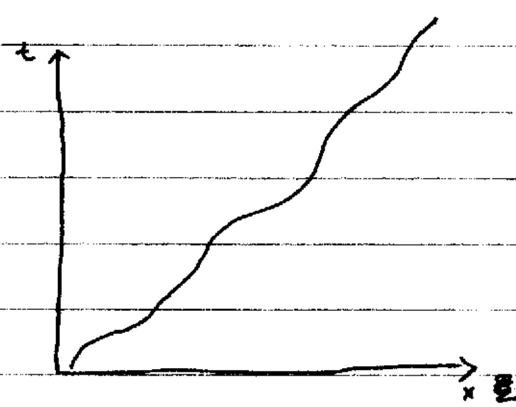
comp. in $\Rightarrow$ tensile back

Also, $+\frac{1}{c} f(L-ct) = +\frac{1}{c} g(L+ct) + \text{const}$

$\therefore$ a forward moving wave is bounced back with it's displacement uniformly shifted (the shift depends on the initial data).

<u>Shocks</u> Solutions where the strain and particle velocity are discontinuous (across moving boundaries) but the displacement is continuous. Then we have to replace the governing equations with "jump conditions"

Suppose we have a <u>shock</u> propagating at $x = s(t)$



Displacement cont: $u(s^+(t), t) = u(s^-(t), t)$

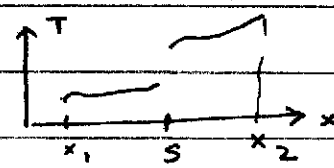
Differentiate with respect to $t$.

$$u_x(s^+(t), t)\,\dot{s} + u_t(s^+(t), t)$$
$$= u_x(s^-(t), t) + u_t(s^-(t), t)$$

① or $\boxed{[\![v]\!] + \dot{s}\,[\![\varepsilon]\!] = 0}$ $\qquad [\![q]\!] = q(s^+) - q(s^-)$

Balance of linear momentum:

$$\frac{d}{dt}\left[\int_{x_1}^{s^-(t)} \rho A u_t \, dx + \int_{s^+(t)}^{x_2} \rho A u_t \, dx\right] = T(x_2) - T(x_1)$$




$$\therefore \cancel{\int_{x_1}^{s^-} \rho A u_{tt} \, dx} + \dot{s}\,\rho A u_t(s^-, t)$$
$$+ \cancel{\int_{s^+}^{x_2} \rho A u_{tt} \, dx} - \dot{s}\,\rho A\, u_t(s^+, t) = \cancel{\int_{x_1}^{s^-} \frac{\partial T}{\partial x} dx} + \cancel{\int_{s^+}^{x_2} \frac{\partial T}{\partial x} dx}$$
$$- T(s^-, t) + T(s^+, t)$$

$$\Rightarrow \quad \boxed{[\![T]\!] + \dot{s}\,[\![\rho A v]\!] = 0}$$

Since $T = EA\,\varepsilon$,

② $\boxed{[\![EA\varepsilon]\!] + \dot{s}\,[\![\rho A v]\!] = 0}$

① + ② are the jump conditions.

Shocks in uniform bars: $\rho, A, E$ continuous.

$\therefore$ ② $\Rightarrow E[\![\varepsilon]\!] + \rho\dot{s}[\![v]\!] = 0$

$\therefore$ ① + ② $\Rightarrow \rho\dot{s}^2[\![\varepsilon]\!] - E[\![\varepsilon]\!] = 0$

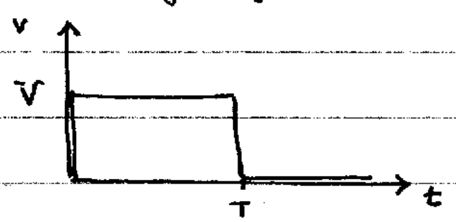
$$\Rightarrow \rho\dot{s}^2 = E \quad \text{or} \quad \boxed{\dot{s}^2 = \frac{E}{\rho} = c^2}$$

$\therefore$ Shocks propagate with the wave speed.

$\therefore$ we can take $f, g$ to be continuous but with discont. first derivatives and enforce the jump conditions.

<u>Example 1.</u> <u>Impact problem</u>.

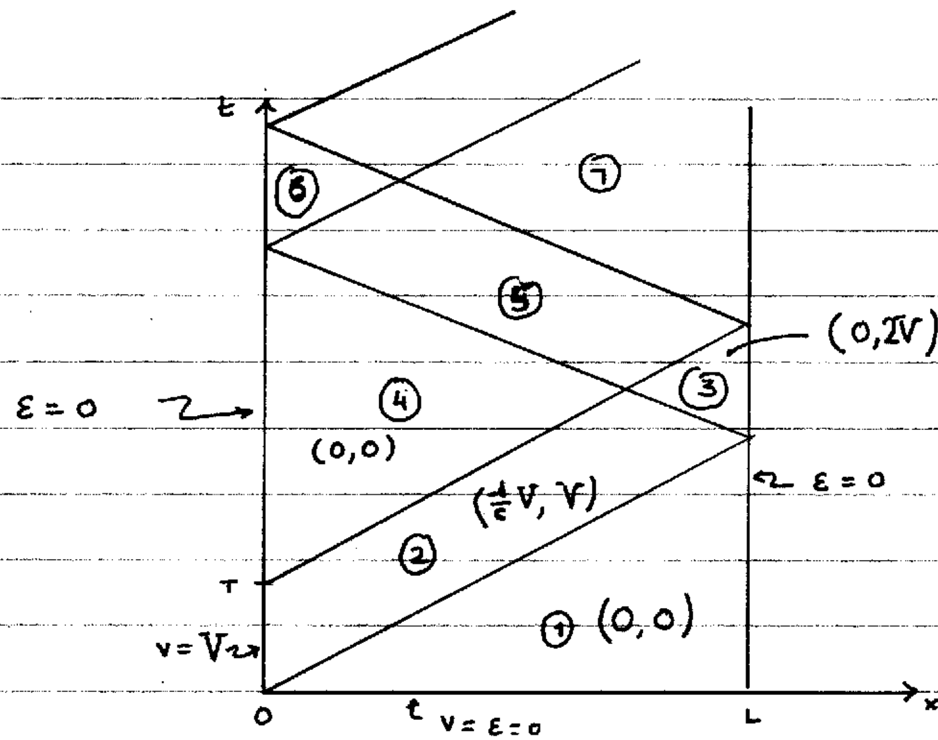
Consider a bar of length $L$ initially at rest with no applied forces, subjected to a velocity pulse at $x=0$ as shown.




$$v = \begin{cases} V & 0<t<T \\ 0 & t>T \end{cases}$$

Since the initial and boundary conditions are piecewise $\overbrace{\text{constant}}^{\text{in } v, \varepsilon}$, we look for piecewise constant solutions.

By looking at the characteristics, we infer that it has the form shown in the figure.




Region 1: $v = \varepsilon = 0$

Region 2: $v = V$ (from b.c at $x = 0$)

To find $\varepsilon$, look at the jump condition at the 1, 2 boundary:

$$[\![ v ]\!] + \dot{s}\, [\![ \varepsilon ]\!] = 0$$

$$(0 - V) + c\,(0 - \varepsilon) = 0 \quad \Rightarrow \quad \varepsilon = -\frac{1}{c} V$$

Region 3: $\varepsilon = 0$

We can find $v$ two ways:

First, jump condition at 2, 3 boundary

$$(v - V) - c\left(0 - \left(-\frac{1}{c} V\right)\right) = 0$$

$$\Rightarrow \quad v = 2V$$

Alternately, from our study of free boundary conditions, we know the "returning wave" has the opposite strain and same particle velocity. So region 5 is given by $(\frac{1}{c}V, V)$
Region 3 is a superposition of regions 2 and 5. So in region 3

$$(\varepsilon, v) = \left(-\frac{1}{c}V,\ V\right) + \left(\frac{1}{c}V,\ V\right) = (0, 2V)$$

Region 4: $v = \varepsilon = 0$

either by inspection of by a combination of b.c and 2/4 j.c.

Region 5: $(\varepsilon, v) = (\frac{1}{c}V,\ V)$

using the argument above, or using the 4/5 and 3/5 j.c.

Exercises

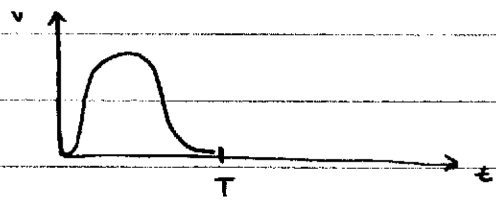
(a) Plot the total linear momentum as a function of time

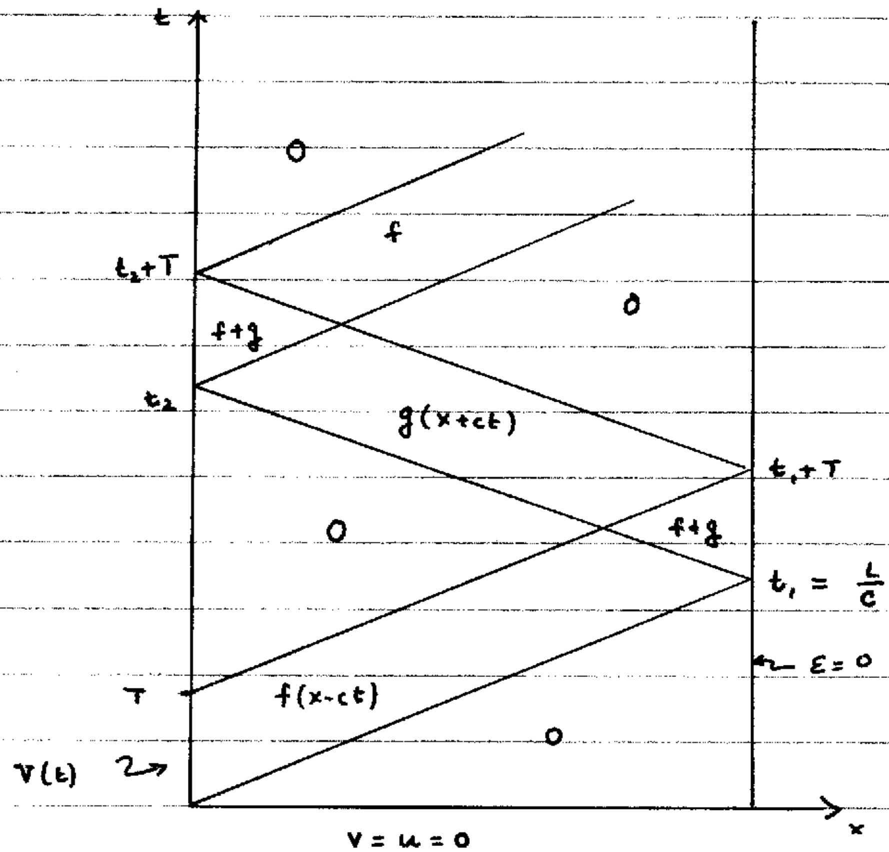
(b) Plot the total kinetic energy as a function of time $K.E = \rho A \int_0^L \frac{1}{2} v^2 dx$

~~(c) Plot~~ the total potential energy $P.E = \rho E \int_0^L \frac{1}{2} \varepsilon^2 dx$

(d) Plot the total energy $K.E + P.E$

Example 2. Same as example 1, but with general pulse




We have to find $f, g$ on the intervals $(-cT, 0)$, $(ct_1, c(t_1+T))$ resp.

At $x=0$, $\dot u(0,t) = V(t)$, $u(0,0) = 0$

$$\Rightarrow u(0,t) = \int_0^t V(\tau)\,d\tau = f(-ct)$$

$$\therefore f(-\xi) = +\int_0^{\xi/c} V(\tau)\,d\tau \qquad \xi \in (0, cT)$$

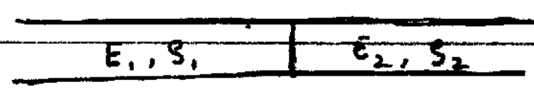
Similarly $\quad x = L, \quad t \in (t_1, t_1 + T) \qquad t_1 = L/c$

$$g'(L+ct) + f'(L-ct) = 0 \quad \Rightarrow g$$

$$g(L+ct_1) = f(L-ct_1) = 0$$

Continue similarly for longer times.

## Example 3. Transmission problem




Consider a bar of uniform cross-section A, but made of heterogeneous materials,

$$E = \begin{cases} E_1 & x < 0 \\ E_2 & x > 0 \end{cases} \qquad \rho = \begin{cases} \rho_1 & x < 0 \\ \rho_2 & x > 0 \end{cases}$$

Suppose a wave $f(x - c_1 t)$ travels through material 1 and hits the interface. Part of it is transmitted $h(x - c_2 t)$ while part of it it reflected $g(x + c_1 t)$

To find $g, h$, we use the jump conditions

$$[\![ v ]\!] = 0$$
$$[\![ T ]\!] = 0$$

since the interface $x = 0$ does not move $(\dot{s} = 0)$.

$$\therefore \quad -c_1 f'(-c_1 t) + c_1 g'(+c_1 t) = -c_2 h'(-c_2 t)$$

$$E_1 f'(-c_1 t) + E_1 g'(c_1 t) = E_2 h'(-c_2 t)$$

.... solve for $g'(c_1 t)$, $h'(c_2 , t)$
use the initial data to
find $g$, $h$.

Infact,

$$h'(-c_2 t) = \frac{E_1 c_2 + E_2 c_1}{2 E_1 c_1} f'(-c_1 t)$$

and $$g'(c_1 t) = \frac{E_2 c_1 - E_1 c_2}{2 E_1 c_1} h'(-c_2 t)$$

Note : If $\boxed{E_1 c_2 = E_2 c_1}$, then $h'(-c_2 t) = f'(-c_1 t)$

and $g'(c_1 t) = 0$

i.e., the wave is transmitted undistorted
(but at a different speed)
with no reflection.

$$E_1 c_2 = E_2 c_1 \iff \frac{E_1}{c_1} = \frac{E_2}{c_2} \quad \left(\begin{array}{c}\text{impedance} \\ \text{matched}\end{array}\right)$$

$$\boxed{\frac{E}{c} = \rho c \quad \ldots \text{ impedance}}$$