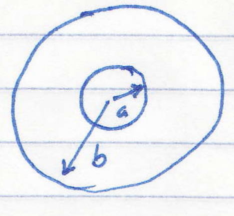
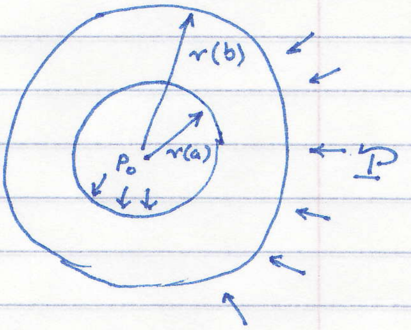
Radial deformation of a neo-Hookean spherical shell





$\Omega = \{ \underset{\sim}{x} : \; a < |\underset{\sim}{x}| < b \}$ reference conf.

<u>Assume</u> deformation of the form

$$r = r(R)$$
$$\theta = \Theta \qquad \text{(radial deformation)}$$
$$\phi = \Phi$$

$$\nabla \underset{\sim}{y} = \begin{pmatrix} r' & & \\ & r/R & \\ & & r/R \end{pmatrix} \quad \text{in spherical coordinates}$$

$$\underset{\sim}{B} = \begin{pmatrix} (r')^2 & & \\ & r^2/R^2 & \\ & & r^2/R^2 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{T} = \begin{pmatrix} -p + 2\alpha (r')^2 & & \\ & -p + 2\alpha \left(\frac{r}{R}\right)^2 & \\ & & -p + 2\alpha \left(\frac{r}{R}\right)^2 \end{pmatrix} \qquad p = p(r)$$

Incompressibility: $\det F = 1 \iff r' \frac{r^2}{R^2} = 1 \iff \boxed{r^3 = R^3 + c}$ $\boxed{r' = \frac{R^2}{r^2}}$

Equilibrium: $\operatorname{div} T = 0$ reduces to

$$\frac{dT_{rr}}{dr} + \frac{2}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0$$

Integrate from $r(a)$ to $r(b)$

$$T_{rr}(r(b)) - T_{rr}(r(a)) = -\int_{r(a)}^{r(b)} \frac{2\,(T_{rr} - T_{\theta\theta})}{r}\, dr$$

or $$-P + p_0 = -4\alpha \int_{r(a)}^{r(b)} \frac{1}{r}\left(\left(\frac{R}{r}\right)^4 - \left(\frac{r}{R}\right)^2\right) dr$$

Note $r' = \frac{R^2}{r^2}$ due to incompressibility

Change variables: $r \mapsto R$, $dr = \frac{dr}{dR}\, dR = \frac{R^2}{r^2}\, dr$

$$\therefore -P + p_0 = -4\alpha \int_a^b \frac{R^6 - r^6}{r^7}\, dR$$

$$= +4\alpha \int_a^b \frac{(R^3+c)^2 - R^6}{(R^3+c)^{7/3}}\, dR$$

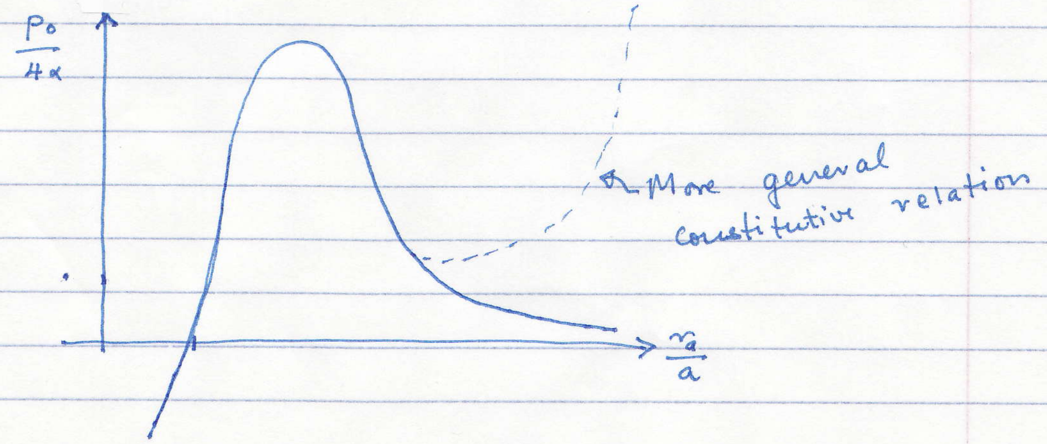
$$= 4\alpha \left[\frac{5R^4 + 4Rc}{4(R^3+c)^{4/3}}\right]_a^b = 4\alpha\left[\frac{5R^4 + 4Rc}{4[r(R)]^4}\right]_a^b$$

$$\therefore \boxed{-P + p_0 = 4\alpha \left[\frac{5R^4 + 4Rc}{4[r(R)]^4}\right]_a^b} \quad r^3 = R^3 + c \qquad (1)$$

Spherical balloon

$P = 0$, set $c = a^3 - r_a^3$, $r(b) = \left(b^3 + a^3 - r_a\right)^3$

(1) gives a relation btn p_0 vs r_a



<u>Cavitation</u> Subject a block to triaxial extension

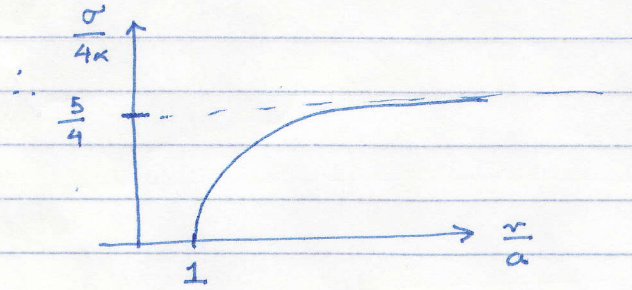
$$-P = \sigma > 0, \quad p_0 = 0 \qquad b \to \infty$$

$$\sigma = 4\alpha \left[\frac{R^4 + 4R\left(r(R)\right)^3}{4\left(r(R)^4\right)} \right]_a^b = 4\alpha \left[\frac{1}{4}\left(\frac{R}{r(R)}\right)^4 + \frac{R}{r(R)} \right]_a^b$$

As $b \to \infty \qquad \frac{r}{R} \to 1$

$$\frac{\sigma}{4\alpha} = \frac{5}{4} - \underbrace{\left(\frac{1}{4}\left(\frac{a}{r(a)}\right)^4 + \frac{a}{r(a)} \right)}$$

Notice that this is always <1 for applied tension



Can not sustain stress larger than $\frac{5}{4} \cdot 4\alpha$!

In[1]:=
```
r[x_] := (x^3 + c)^(1/3)
r[a]
r[b]
```

Out[2]= $\left(a^3 + c\right)^{1/3}$

Out[3]= $\left(b^3 + c\right)^{1/3}$

In[4]:=
```
f[x_] := (5 * x^4 + 4 * x * c) / (4 * r[x]^4)
f[b]
```

Out[5]= $$\frac{5\,b^4 + 4\,b\,c}{4\,\left(b^3 + c\right)^{4/3}}$$

In[6]:=
```
p = (f[b] - f[a]) /. c → (r^3 - a^3)
```

Out[6]= $$-\frac{5\,a^4 + 4\,a\,\left(-a^3 + r^3\right)}{4\,\left(r^3\right)^{4/3}} + \frac{5\,b^4 + 4\,b\,\left(-a^3 + r^3\right)}{4\,\left(-a^3 + b^3 + r^3\right)^{4/3}}$$

In[7]:=
```
pressure = p /. {a → 1, b → 1.1}
```

Out[7]= $$-\frac{5 + 4\,\left(-1 + r^3\right)}{4\,\left(r^3\right)^{4/3}} + \frac{7.3205 + 4.4\,\left(-1 + r^3\right)}{4\,\left(0.331 + r^3\right)^{4/3}}$$

In[8]:=
```
Plot[pressure, {r, 1, 10}, AxesOrigin → {0, 0}]
```

Out[8]=