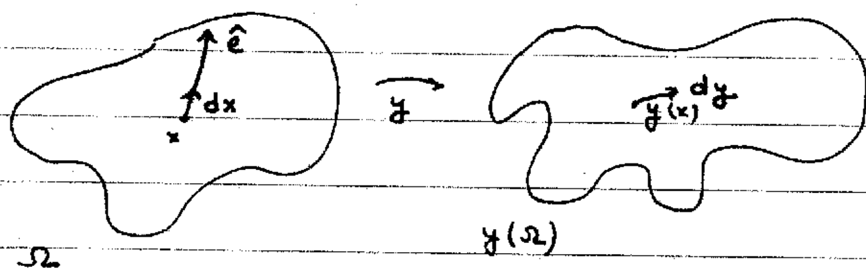
Rubber elasticity : Finite deformation theory.

Recall:



Deformation: $y : \Omega \to \mathbb{R}^3$

Deformation gradient : $F = Dy$; $F_{ij} = \dfrac{\partial y_i}{\partial x_j}$ $i,j = 1, 2, 3$

Cauchy Green matrix : $C = F^T F$

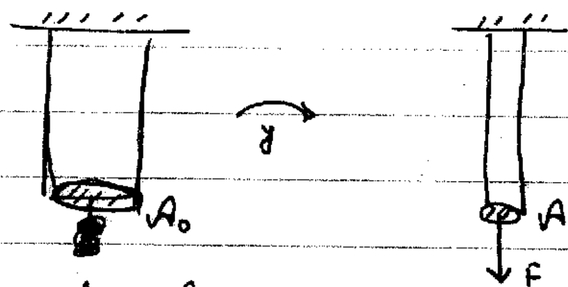
Strech in direction $\hat{e}$ $= \dfrac{|dy|}{|dx|} = \sqrt{\hat{e} \cdot C\hat{e}}$

Change in angle btn 2 perpendicular directions $\hat{e}_1$ and $\hat{e}_2$ $= \text{Arc sin}\left(\dfrac{\hat{e}_1 \cdot C\hat{e}_2}{\sqrt{(\hat{e}_1 \cdot C\hat{e}_1)}\sqrt{(\hat{e}_2 \cdot C\hat{e}_2)}}\right)$

$$= \text{Arc sin}\left(\frac{\hat{e}_2 \cdot C\hat{e}_1}{\sqrt{(\hat{e}_1 \cdot C\hat{e}_1)(\hat{e}_2 \cdot C\hat{e}_2)}}\right)$$

There are two notions of stress in finite elasticity

Basic idea



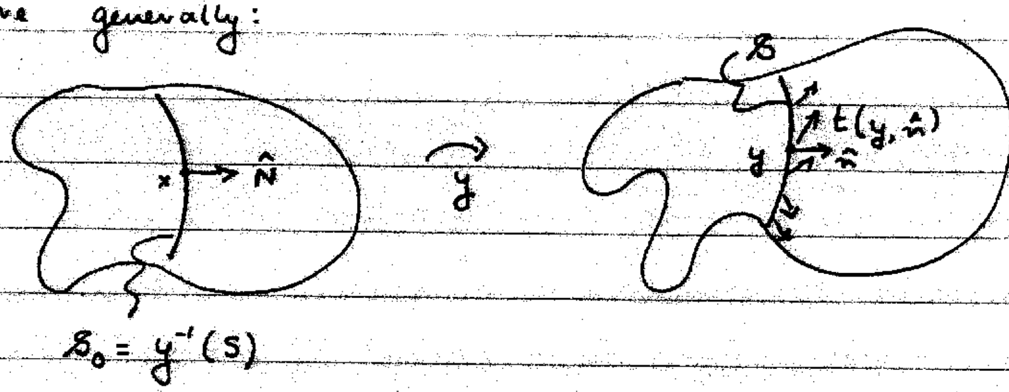
ref. conf.

Suppose we take a bar with cross-section A_0 and apply a force F to it. The bar elongates, but the area also changes. So we can calculate stress in 2 ways:

1) $\frac{F}{A}$ ... "true stress"

2) $\frac{F}{A_0}$ ... "engineering or nominal stress".

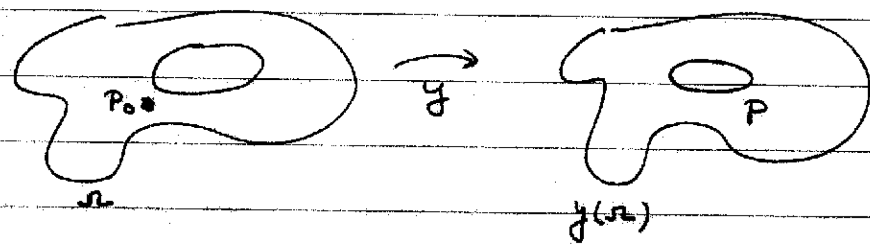
More generally:



Traction (force per unit current area) on the surface $\mathcal{S}$ ~~per~~ in deformed configuration $\Big\}$ $t(y, \underset{\uparrow}{\hat{n}})$ normal $= \mathcal{S}$

But surface $\mathcal{S}$ corresponds to a surface $\mathcal{S}_0$ in reference configuration. We can ~~simply~~ choose to express the traction as force per unit reference area ... $S(x, \underset{\uparrow}{\hat{N}})$ normal to $\mathcal{S}_0$

$\therefore$



$P = y(P_0)$

or

$P_0 = y^{-1}(P)$

Total force on a part of the body that occupies P_0 in the reference configuration (or equivalently P in the deformed configuration

$$\int_P \underset{\uparrow}{b(y)}\, dV_y + \int_{\partial P} t(y, \hat{n})\, dA_y$$

Body force per unit def. vol.

$$= \int_{P_0} \underset{\uparrow}{B(x)}\, dV_y + \int_{\partial P_0} S(x, \hat{N})\, dA_x$$

Body force per unit ref. volume

Total moment on part of the body ...

$$\int_{\partial P} (y-c) \times t(y,\hat{n})\, dA_y + \int_{P} (y-c) \times b(y)\, dV_y$$

$$= \int_{\partial P_0} (y(x)-c) \times s(x,\hat{N})\, dA_x + \int_{P_0} (y(x)-c)\, B(x)\, dV_y$$

Balance of Forces:

$$t(y,\hat{n}) = T(y)\hat{n}$$

↑ Cauchy Stress matrix

$$s(x,\hat{N}) = S(x)\hat{N}$$

↑ Piola-Kirchhoff Stress matrix

or

nominal stress matrix

Can show: $S(x) = \left(\det F(x)\right) T(y(x))\, F^{-T}(x)$

or

$$S = (\det F)\, T\, F^{-T}$$

$\mathrm{Div}_y\, T + b = 0$ $\qquad \dfrac{\partial T_{ij}}{\partial y_j} + b = 0$ on $y(\Omega)$

or

$\mathrm{Div}_x\, S + B = 0$ $\qquad \dfrac{\partial S_{ij}}{\partial x_j} + B = 0$ on Ω

Balance of moments:

$$T = T^T$$

← Cauchy stress is symmetric

or

$$SF^T = FS^T$$

← Note that Piola-Kirchhoff stress is not necessarily symmetric.

Constitutive relations: $T = T(F)$

or

$S = S(F)$

or

stored energy density $W = W(F)$

and

$$T = \frac{1}{\det F}\left(\frac{\partial W}{\partial F}\right)F^T \quad \text{or} \quad S = \frac{\partial W}{\partial F}$$

$$T_{ij} = \frac{1}{(\det F)}\left(\frac{\partial W}{\partial F_{ik}}\right)F_{jk} \quad \text{or} \quad S_{ij} = \frac{\partial W}{\partial F_{ij}}$$

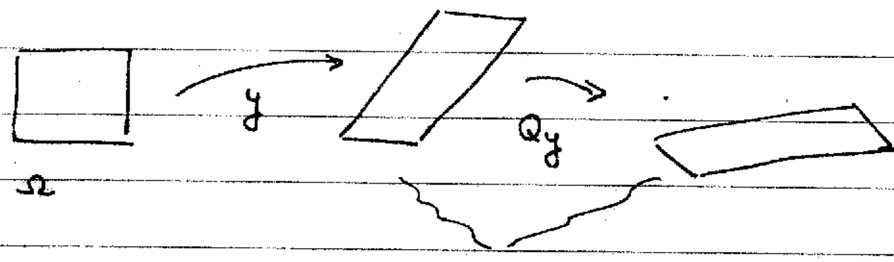
"Hyperelasticity".

Constitutive relations must satisfy two restrictions

1. Frame-indifference

$$W(QF) = W(F) \quad \forall \quad Q \text{ rotations}$$

"rigid body rotations do not change energy"



these states should have same energy.

$$\Leftrightarrow \quad W(F) = \tilde{W}(C) \qquad C = F^T F$$

2. Material symmetry

$$W(FR) = W(F) \quad \forall \quad R \in P \ldots \text{ point group}$$

"some directions in the ref. conf. are identical"

Eg: Isotropy: $W(FR) = W(F) \;\; \forall$ rotations R

$$\Leftrightarrow \quad W(F) = \hat{W}(B) \qquad B = FF^T$$

<u>Internal constraints</u>: Some materials have internal constraints.

Eg: 1. rubber is incompressible $\Rightarrow$ $\det F = 1$

2. fiber reinforced composites are inextensible in the fiber direction $\hat{e}$ $\Big\} \Rightarrow \hat{e} \cdot F^T F \hat{e} = 1$.

In such cases, we have to add an indeterminate constraint stress

Eg: incompressibility $\Rightarrow$ $\underset{\sim}{T} = -\underset{\substack{\uparrow \\ \text{indeterminate} \\ \text{constant}}}{p} \underset{\sim}{I} + \underset{\sim}{T}(F)$

<u>Constitutive models for rubber</u>:

1. neo-Hookean : $W = \alpha \left(\operatorname{tr} FF^T - 3 \right) = \alpha \left(\operatorname{tr} F^T F - 3 \right)$

(as in the statistical derivation)

$$\underset{\sim}{T} = -p \underset{\sim}{I} + 2\alpha \underset{\sim}{B} \qquad B = FF^T$$

2. Mooney-Rivlin: $\cancel{W = \alpha(\mathrm{tr}\, FF^T - 3) + \alpha}$

$$W = \alpha_1 (I_1 - 3) + \alpha_2 (II_2 - 3)$$

$B = FF^T$
$C = F^T F$

where $I_1 = \mathrm{tr}\, B = \mathrm{tr}\, C$

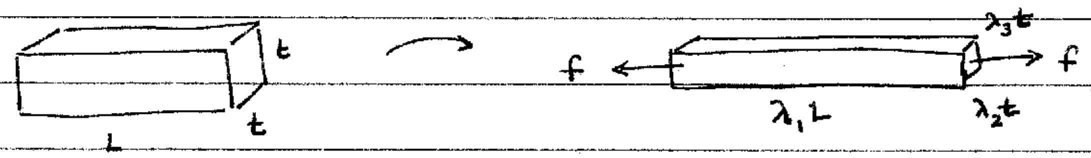
$$II_2 = \frac{1}{2} \left((\mathrm{tr}\, B)^2 - \mathrm{tr}\, B^2 \right)$$

$$= \frac{1}{2} \left((\mathrm{tr}\, C)^2 - \mathrm{tr}\, C^2 \right)$$

or $\underset{\sim}{T} = -p \underset{\sim}{I} + 2\alpha_1 B - 2\alpha_2 B^{-1}$ $\qquad B = FF^T$

Uniaxial extension of a neo-Hookean bar

$\Omega = (0,L) \times (0,t) \times (0,t)$ subjected to a total force f on the faces $\{x_1 = 0\}$, $\{x_1 = L\}$



Look for a solution of the type $y_i = \lambda_i x_i \quad i = 1,2,3$

$\lambda_i > 0$

$$F = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

Incompressibility : $\lambda_1 \lambda_2 \lambda_3 = 1$

Assume $\lambda_2 = \lambda_3$ (one has to be careful in nonlinear problems due to bifurcations. It turns out that this can be justified here).

$$\Rightarrow \lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = \lambda^{-1/2}$$

$$\Rightarrow F = \begin{pmatrix} \lambda & & \\ & \lambda^{-1/2} & \\ & & \lambda^{-1/2} \end{pmatrix}$$

$$B = FF^T = \begin{pmatrix} \lambda^2 & & \\ & \lambda^{-1} & \\ & & \lambda^{-1} \end{pmatrix}$$

$$\Rightarrow \quad T = -p\,I + 2\alpha B = \begin{pmatrix} -p + 2\alpha\lambda^2 & & \\ & -p + 2\alpha\lambda^{-1} & \\ & & -p + 2\alpha\lambda^{-1} \end{pmatrix}$$

Equilibrium: $\quad \operatorname{div}_y T = 0 \quad \Rightarrow \quad \operatorname{grad}_y p = 0$

$$\Rightarrow \quad p = \text{constant}$$

Boundary conditions:

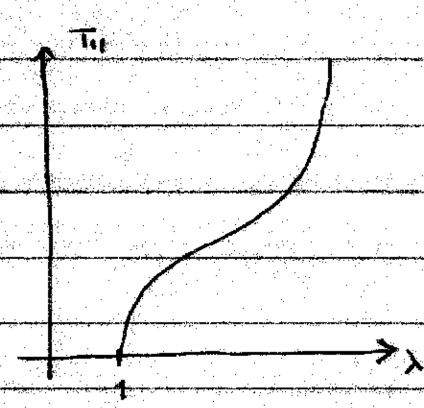
$$T_{11}\,(\lambda_2 t)(\lambda_3 t) = f$$

$$T_{12} = T_{23} = T_{13} = 0 \quad \leftarrow \text{automatic by our choice of } y.$$

$$T_{22} = T_{33} = 0$$

$$\Rightarrow \begin{cases} f = \dfrac{T_{11}\,t^2}{\lambda} = \dfrac{t^2}{\lambda}\left(-p + 2\alpha\lambda^2\right) \\[2ex] 0 = -p + 2\alpha\lambda^{-1} \qquad \Rightarrow \boxed{p = 2\alpha\lambda^{-1}} \end{cases}$$

$$\Rightarrow \begin{cases} T_{11} = 2\alpha\left(\lambda^2 - \lambda^{-1}\right) \\ \quad \text{or} \\ f = \dfrac{2\alpha t^2}{\lambda}\left(\lambda^2 - \lambda^{-1}\right) \end{cases}$$



Note that the stress-stretch relation is <u>non-linear</u>.

If we linearize about $\lambda = 1$, or $\lambda = 1+\varepsilon$,

$$T_{11} \doteq 2\alpha\left(1+2\varepsilon - (1-\varepsilon)\right) = \underbrace{6\alpha}_{\text{Young's modulus.}}\,\varepsilon$$

lateral contraction: $\lambda_2 - 1 = \frac{1}{\sqrt{\lambda}} - 1 = 1 - \frac{1}{2}\varepsilon - 1$

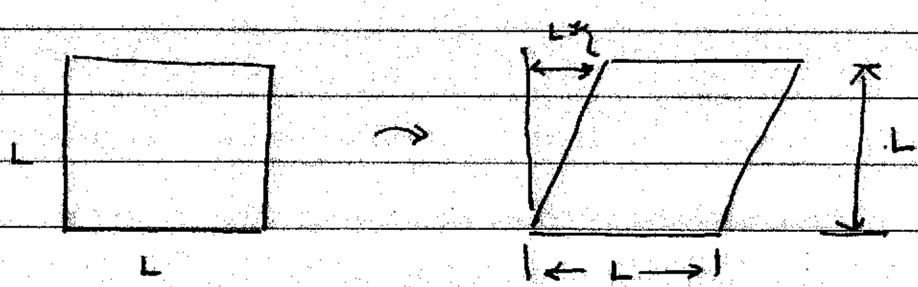
$$= -\underbrace{\tfrac{1}{2}}_{\text{Poisson's ratio}}\,\varepsilon$$

(the value $1/2$ reflects incompressibility).

Simple shear of a neo-Hookean plate

$\Omega = (0,L)\times(0,L)\times(0,t)$ undergoing simple shear

$$y_1 = x_1 + \kappa x_2, \quad y_2 = x_2, \quad y_3 = x_3$$



$$F = \begin{pmatrix} 1 & \kappa & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1+\kappa^2 & \kappa & \\ & 1 & \\ \text{sym} & & 1 \end{pmatrix}$$

$$\Rightarrow \quad T = \begin{pmatrix} -p + 2\alpha(1+\kappa^2) & 2\alpha\kappa & 0 \\ & -p + 2\alpha & 0 \\ \text{sym} & & -p + 2\alpha \end{pmatrix}$$

Equilibrium $\Rightarrow$ $p = \text{constant}$ as before.

Note: 1. $T_{11} - T_{22} = 2\alpha\kappa^2 = \kappa T_{12}$

or

$$T_{11} - T_{22} = \kappa T_{12}$$

Each of the quantities in this equation can be measured independently. In other words, the ~~theory~~ equation has no constitutive or fitting parameters. Such relations are useful in verifying the validity of the theory and are known as universal relations.

2. $T_{11} - T_{22} \neq 0$ if $\kappa \neq 0$

Poynting effect

3. $T_{12} = 2\alpha\kappa$

Linear shear stress – shear strain relation.

3. Triaxial extension of a neo-Hookean block.

(Rivlin buckling) [Rivlin, R.S, Quart. J. Appl. Math 32 (1974)]

$\Omega = [0,1]^3$

$y_i = \lambda_i x_i$ $\quad i$ not summed.

Assume that $\lambda_i > 0$.

$\therefore$ for a neo-Hookean material,

$T_{ii} = -\pi + 2\alpha \lambda_i^2$ $\quad i$ not summed

$p = \pi$!

$T_{ij} = 0 \quad i \neq j$

$\lambda_1 \lambda_2 \lambda_3 = 1$.

The equilibrium equation, $\operatorname{div} T = 0 \implies \pi =$ arbitrary const

Now, the force on the i^{th} face of the block

$$\tau_i = T_{ii} \lambda_j \lambda_k \qquad i \neq j \neq k \neq i.$$

Suppose $\tau_1 = \tau_2 = \tau_2 = \tau$. i.e., the forces on all faces of the block are equal.

$\therefore$ the we need to solve the following equations

$$\tau = -\pi \lambda_2 \lambda_3 + 2\alpha \lambda_1^2 \lambda_2 \lambda_3 \qquad (1)$$

$$\tau = -\pi \lambda_1 \lambda_3 + 2\alpha \lambda_2^2 \lambda_1 \lambda_3 \qquad (2)$$

$$\tau = -\pi \lambda_1 \lambda_2 + 2\alpha \lambda_3^2 \lambda_1 \lambda_2 \qquad (3)$$

$$\lambda_1 \lambda_2 \lambda_3 = 1 \qquad (4)$$

for λ_i and π given τ and α.

Divide (1), (2) and (3) by 2α and set $\sigma = \frac{\tau}{2\alpha}$ *

Also use (4) to eliminate λ_3 from (1), (2) and (3).

$$\sigma = -\left(\frac{\pi}{2\alpha}\right)\frac{1}{\lambda_1} + \lambda_1$$

$$\sigma = -\left(\frac{\pi}{2\alpha}\right)\frac{1}{\lambda_2} + \lambda_2$$

$$\sigma = -\left(\frac{\pi}{2\alpha}\right)\lambda_1\lambda_2 + \frac{1}{\lambda_1\lambda_2} \Rightarrow \left(\frac{\pi}{2\alpha}\right) = \frac{1}{\lambda_1\lambda_2}\left(\frac{1}{\lambda_1\lambda_2} - \sigma\right)$$

$$\therefore \quad \sigma = -\frac{1}{\lambda_1^2\lambda_2}\left(\frac{1}{\lambda_1\lambda_2} - \sigma\right) + \lambda_1 \qquad -(5)$$

$$\sigma = -\frac{1}{\lambda_1\lambda_2^2}\left(\frac{1}{\lambda_1\lambda_2} - \sigma\right) + \lambda_2 \qquad -(6)$$

Multiply (5) by λ_1^3 and (6) by λ_2^3 and factor them to obtain

$$\left(\lambda_1^2 - \frac{1}{\lambda_2}\right)\left(\lambda_1^2 - \sigma\lambda_1 + \frac{1}{\lambda_2}\right) = 0$$

$$\left(\lambda_2^2 - \frac{1}{\lambda_1}\right)\left(\lambda_2^2 - \sigma\lambda_2 + \frac{1}{\lambda_1}\right) = 0$$

$\therefore$ (7a) $\lambda_1^2 - \frac{1}{\lambda_2} = 0$ or (7b) $\lambda_1^2 - \sigma\lambda_1 + \frac{1}{\lambda_2} = 0$

(8a) $\lambda_2^2 - \frac{1}{\lambda_1} = 0$ or (8b) $\lambda_2^2 - \sigma\lambda_2 + \frac{1}{\lambda_1} = 0$

$\therefore$ We have many cases

1. 7a and 8a hold
2. 7a and 8b hold
3. 7b and 8a hold
4. 7b and 8b hold.

* Note: σ is a normalized force.

Case 1. $\lambda_1^2 - \frac{1}{\lambda_2} = 0$. $\lambda_2^2 - \frac{1}{\lambda_1} = 0$

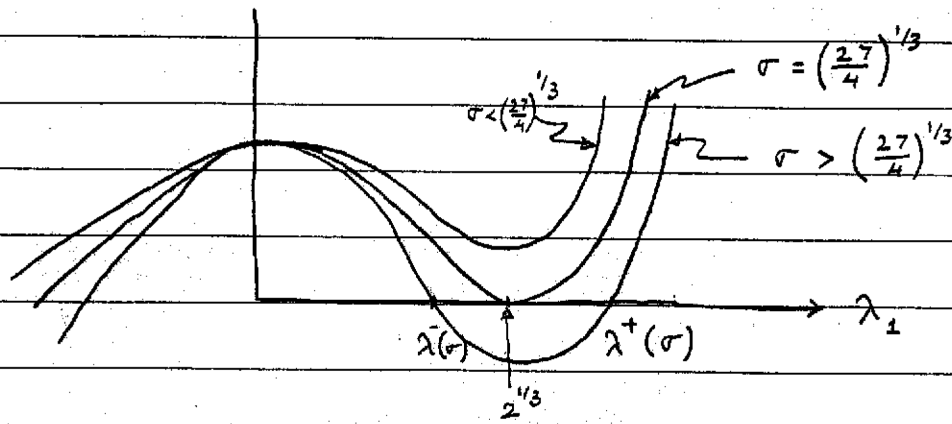
$$\Rightarrow \lambda_1 = \lambda_2 = 1 \Rightarrow \lambda_3 = 1 \qquad - \text{(Soln A)}$$

Case 2. $\lambda_1^2 - \frac{1}{\lambda_2} = 0$. $\lambda_2^2 - \sigma\lambda_2 + \frac{1}{\lambda_1} = 0$

Eliminating λ_2, $\lambda_1^3 - \sigma\lambda_1^2 + 1 = 0$

We need to solve this cubic for given $\sigma > 0$

Plot the cubic for given σ.



$\therefore$ (i) $0 \leq \sigma < \left(\frac{27}{4}\right)^{1/3}$: No positive root.

(ii) $\sigma = (27/4)^{1/3}$: One positive double root:

$$\left.\begin{aligned} &\lambda_1 = 2^{1/3} \\ &\therefore \lambda_2 = \frac{1}{\lambda_1^2} = 2^{-2/3} \qquad \lambda_3 = 2^{1/3} \end{aligned}\right] \text{(Soln B)}$$

(iii) $\sigma > (27/4)^{1/3}$: Two positive roots – $\lambda^+(\sigma)$ and $\lambda^-(\sigma)$

where $0 < \lambda^-(\sigma) < 2^{1/3} < \lambda^+(\sigma)$

and $\lambda^- \downarrow$ and $\lambda^+ \uparrow$ as $\sigma \nearrow$.

$$\therefore \lambda_1 = \lambda^- \Rightarrow \lambda_2 = \frac{1}{(\lambda^-)^2} \Rightarrow \lambda_3 = \lambda^- \qquad \text{(soln C)}$$

$$\lambda_1 = \lambda^+ \Rightarrow \lambda_2 = \frac{1}{(\lambda^+)^2} \Rightarrow \lambda_3 = \lambda^+ \qquad \text{(Soln D)}$$

Case 3. Interchange λ_1 and λ_2 in case 2

Case 4. Eliminate λ_2 to obtain $\lambda_1^3 - \sigma\lambda_1^2 + 1 = 0$
which is the same as before.

Thus, the equilibrium solns may be described as follows.

1. Small force - $\sigma < \left(\frac{27}{4}\right)^{1/3}$: Unique soln - unit cube remains undistorted (soln A)

2. Intermediate special force - $\sigma = \left(\frac{27}{4}\right)^{1/3}$: 2 solns
   (a) unit cube remains undistorted (soln A) and
   (b) unit cube deforms to a square plate of thickness $2^{-2/3}$ and side $2^{1/3}$ (soln B).*

3. Large force - $\sigma > \left(\frac{27}{4}\right)^{1/3}$: 3 solns
   (a) unit cube remains undistorted (soln A)
   (b) unit cube deforms to a square plate of thickness $(\lambda^+)^{-2}$ and side λ^+ (soln D)*
   (c) unit cube deforms to a rod of cross-section $\lambda^- \times \lambda^-$ and length $(\lambda^-)^{-2}$. (soln C)*

* Note! Any face of the cube may become the face of the plate or rod.

Remark: In problems where we have multiple equilibrium solution, it is possible at times to obtain uniqueness by considering the "stability" of the equilibrium which is based energy considerations. Equilibrium solns are stationary points of the total potential energy in the system while stable equilibrium solns are the minimizers of the total potential energy.

The total potential energy of our system

$$\xi = \begin{pmatrix} \text{stored elastic energy} \\ \text{in the block} \end{pmatrix} - \begin{pmatrix} \text{work done by} \\ \text{the external forces} \end{pmatrix}$$

$$= \int_\Omega \varphi(F)\, dv_x - \tau(\lambda_1 + \lambda_2 + \lambda_3)$$

$$= \int_\Omega \alpha[\lambda_1 + \lambda_2 + \lambda_3 - 3]\, dv_x - \tau(\lambda_1 + \lambda_2 + \lambda_3)$$

$$= 2\alpha \left\{ \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} - 3\right) - \sigma\left(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1 \lambda_2}\right) \right\}.$$

$\therefore \quad \xi = \xi(l)$ where $l = \{\lambda_1, \lambda_2\} \in \mathbb{R}^2$.

Given σ, we want to find l that minimizes $\xi(l)$.

First necessary condition: $\left.\frac{d}{d\varepsilon} \mathcal{E}(\ell + \varepsilon a)\right|_{\varepsilon=0} = 0 \quad \forall \quad a \in \mathbb{R}^2.$

$$\Leftrightarrow \quad \frac{\partial \mathcal{E}}{\partial \lambda_1} = \frac{\partial \mathcal{E}}{\partial \lambda_2} = 0$$

This gives us equations (5) and (6), the solns to which are the equilibrium solns to the problem.

Second necessary condition: $\left.\frac{d^2}{d\varepsilon^2} \mathcal{E}(\ell + \varepsilon a)\right|_{\varepsilon=0} \geq 0 \quad \forall \quad a \in \mathbb{R}^2.$

$$\Leftrightarrow \quad \left.\underbrace{\begin{pmatrix} \frac{\partial^2 \mathcal{E}}{\partial \lambda_1^2} & \frac{\partial^2 \mathcal{E}}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial^2 \mathcal{E}}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{E}}{\partial \lambda_2^2} \end{pmatrix}}_{M}\right|_{\ell \text{ satisfies first nec. cond.}} \quad \text{is positive semi-definite.}$$

$$\Leftrightarrow \quad \left[\begin{array}{l} \det M \geq 0 \\ \operatorname{tr} M \geq 0 \end{array}\right.$$

Check stability of soln A: Check $M\big|_{\lambda_1 = \lambda_2 = 1}$ is pos. semi-def.

It turns out that at $\lambda_1 = \lambda_2 = 1$,

$$M_{11} = M_{22} = 4\alpha(2-\sigma) \quad \text{and} \quad M_{12} = 2\alpha(2-\sigma)$$

$$\therefore \quad \operatorname{tr} M = 8\alpha(2-\sigma)$$

$$\det M = 12\alpha^2(2-\sigma)^2$$

$\therefore$ soln A satisfies the second nec. cond. if and only if $\boxed{\sigma \leq 2}$

Similarly check for solns B, C and D.

Summary. Notice $1 < \left(\frac{27}{4}\right)^{1/3} < 2$.

(i) $0 < \sigma < \left(\frac{27}{4}\right)^{1/3}$: undistorted unit cube is the only possible minimizing eq. soln.

(ii) $\sigma > 2$: "plate-like" soln with cross-section $\lambda^+ \times \lambda^+$ and thickness $(\lambda^+)^{-2}$ (soln D) is the only possible min. eq. soln.

(iii) $\left(\frac{27}{4}\right)^{1/3} \leq \sigma \leq 2$: both undistorted and platelike solns are possible equilibrium soln.

In this interval we compare $\mathcal{E}(1,1)$ and $\mathcal{E}(\lambda^+, (\lambda^+)^{-2})$ to find the absolute stability.

soln A turns out to be absolutely stable [i.e, $\mathcal{E}(1,1) < \mathcal{E}(\lambda^+,(\lambda^+)^{-2})$] for small σ and soln D is absolutely stable for large σ.

The exchange of stability $[\mathcal{E}(1,1) = \mathcal{E}(\lambda^+, (\lambda^+)^{-2})]$ takes place at $\sigma^* \simeq 1.9019$.

$\therefore$ the bifurcation diagram looks like: