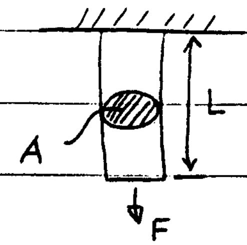
Introduction to Continuum Mechanics

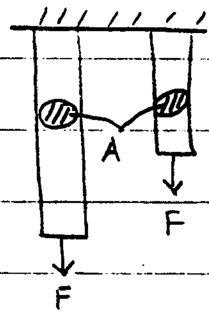
Basic concepts: stress, strain, constitutive relations.

Consider a very simple experiment.
Consider a bar of length L, cross-sectional area A subjected to a force F.



What determines the elongation?

Strain Consider two bars of the same material



and cross-sectional area, but of different length, subjected to the same force F. Clearly, the longer bar elongates more. Therefore, we can only compare

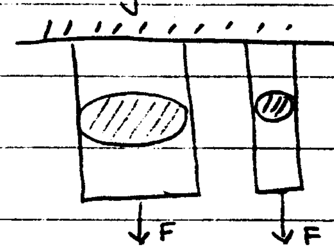
Kinematics

$$\boxed{\frac{\text{change in length}}{\text{original length}} = \text{strain} = \varepsilon}$$

Stress (a) Consider two bars of the same material, length and cross-section subjected to two different forces. Clearly, the bar subjected to the larger force elongates more. Hence force is important.

Balance

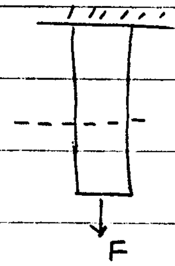
(b) But, it is not all. Consider two bars of the same material and length, but different cross-sections, subjected to the same force F.



Clearly, the thinner bar elongates more. Therefore, we can only compare

$$\boxed{\frac{\text{force}}{\text{area}} = \text{stress} = \sigma}$$

How do we find the stress? We cut a part of the body, draw a free body diagram and balance the forces to find the forces. We then divide by the relevant area.

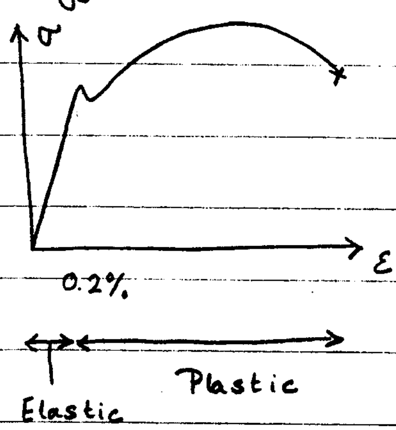


<u>Constitutive Relations</u> Consider two bars of the same length and cross-sectional area, but made of different materials ... steel and rubber. Therefore the material properties are important and they are given by a constitutive relation

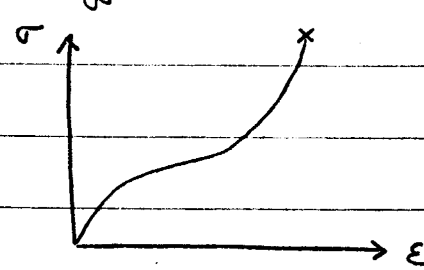
$$\sigma = f(\varepsilon)$$

Example. 1. Hooke's law : $\sigma = E\varepsilon$ E: Young's modulus

2. Typical Steel bar



3. Typical rubber bar



Continuum Mechanics Notice that almost all physics of large quantities of matter - rigid body mechanics, deformable body mechanics, fluid mechanics, heat transfer, elasticity, plasticity, viscoelasticity, ... - follows the same pattern:

I. Kinematics - description of motion.

II. General physical principles or balance laws

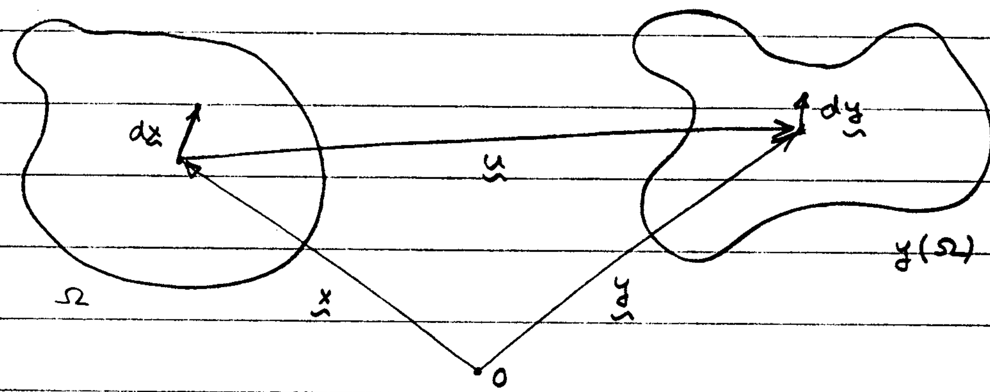
1. Balance of mass
2. Balance of forces or linear momentum
3. Balance of moments or angular momentum
4. Balance of energy (First law of thermodynamics)
5. Entropy inequality (Second law of thermodynamics)

III. Constitutive relations - description of materials properties.

There is much to be learnt by studying all these subjects in an unified framework. This is known as continuum mechanics.

The purpose of this course is to provide a general background in the mechanics of solids. Therefore, we will confine ourselves to only those aspects of continuum mechanics that deal with this. However, we will use the more general framework.

Kinematics



Consider a body that occupies a region Ω in the <u>reference configuration</u>. Consider a typical particle and label it with it's position vector

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

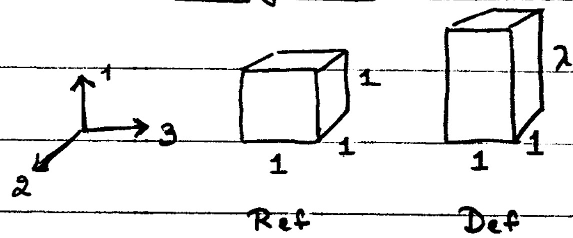
Now deform the body so that this particle goes to

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

We can describe the <u>deformation</u> as the vector valued function $\underset{\sim}{y}(\underset{\sim}{x})$. The body occupies the region $y(\Omega)$ in the <u>deformed configuration</u>. The <u>displacement</u> of the particle is

$$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \underset{\sim}{y} - \underset{\sim}{x}$$

Example 1. Uniaxial deformation

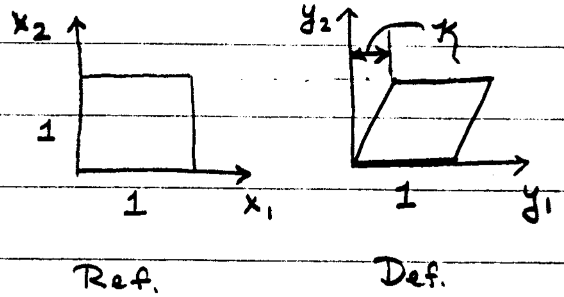


$$\Omega = (0,1) \times (0,1) \times (0,1)$$

$$y_1 = \lambda x_1, \quad y_2 = x_2, \quad y_3 = x_3$$

$\lambda > 1$ extension; $\lambda < 1$ compression.

Example 2. Simple shear



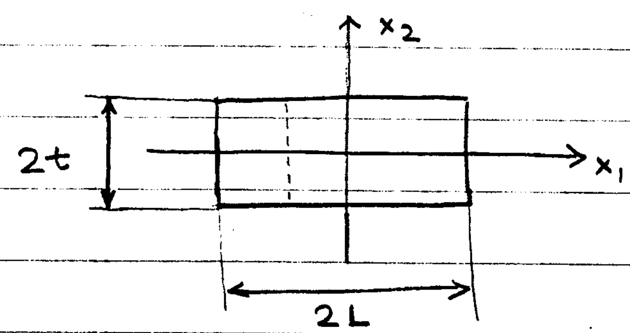
$$\Omega = (0,1) \times (0,1) \times (0,1)$$

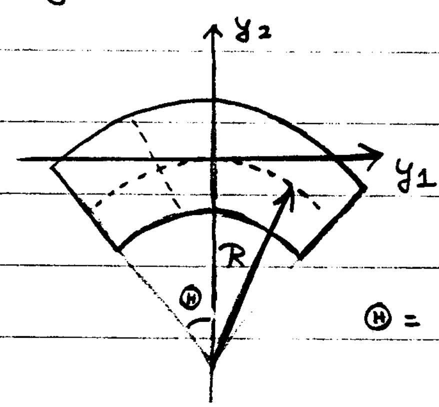
$$y_1 = x_1 + k x_2$$

$$y_2 = x_2$$

$$y_3 = x_3$$

Example 3. Pure bending





$$\Theta = \frac{L}{R}$$

$$\Omega = (-L, L) \times (-t, t) \times (-1, 1)$$

$$y_1 = (R + x_2) \sin\left(\frac{x_1}{R}\right)$$

$$y_2 = (R + x_2) \cos\left(\frac{x_1}{R}\right) - R$$

$$y_3 = x_3$$

The neutral axis $x_2 = 0$ is unstretched and deforms into an arc of radius R. Plane sections remain plane!

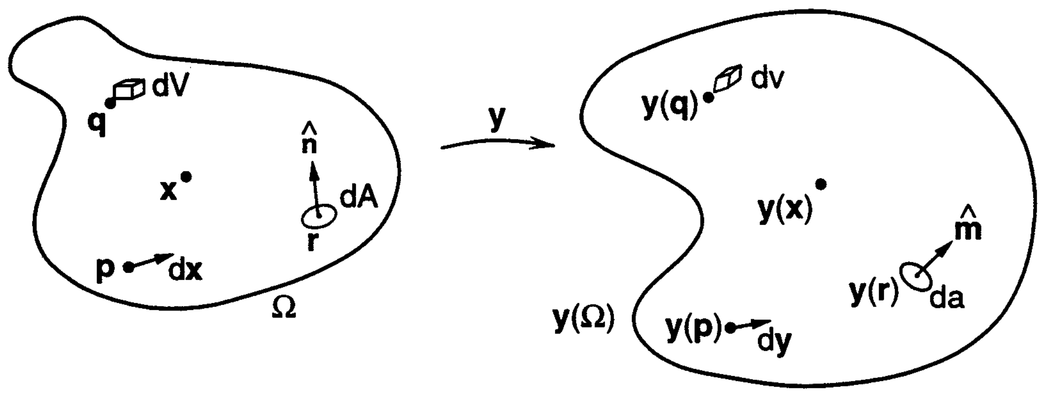


Figure 1: The deformation $\mathbf{y}$ takes the reference configuration on the left to the deformed configuration on the right.

Deformation

Consider a body occupying a region Ω in three-dimensional space $I\!R^3$ as shown in Fig. 1. Let us choose this to be the *reference configuration*; in other words, we use this configuration to describe the body. Let $\mathbf{x} = \{x_1, x_2, x_3\}$ be a typical point in Ω. We call the particle occupying the position $\mathbf{x}$, the particle $\mathbf{x}$. Now, deform the body. The deformation may be described as a function $\mathbf{y} : \Omega \to I\!R^3$ where $\mathbf{y}(\mathbf{x}) = \{y_1(\mathbf{x}), y_2(\mathbf{x}), y_3(\mathbf{x})\}$ denotes position of the particle $\mathbf{x}$ in the *deformed configuration.*

Fig. 2 shows a simple deformation

$$y_1 = (\frac{1}{\sqrt{2}})x_1 - \sqrt{2}x_2 + 3, \qquad y_2 = (\frac{1}{\sqrt{2}})x_1 + \sqrt{2}x_2, \qquad y_3 = x_3. \tag{1}$$

We choose the reference configuration to be an unit cube as shown on the left. Notice that this deformation is planar because $y_3 = x_3$ and we can draw it on a sheet of paper. To completely understand the deformation, we have placed a grid in the reference configuration and followed its deformation. This deformation translates the body to the right, stretches it in the "x_2" direction and then rotates it counter-clockwise by 45^o. Notice that in this deformation, each part of the body has undergone the same distortion; such deformations are known as uniform or *homogeneous* deformations.

Fig. 3 shows another planar deformation,

$$y_1 = x_1 + 0.1\sin(2\pi x_2) + 2, \qquad y_2 = x_2 + 0.1x_1, \qquad y_3 = x_3. \tag{2}$$

of the unit cube. Once again, we follow the deformation of a grid. Notice that in this case, the deformation is not uniform; so we call this an *inhomogeneous* deformation.

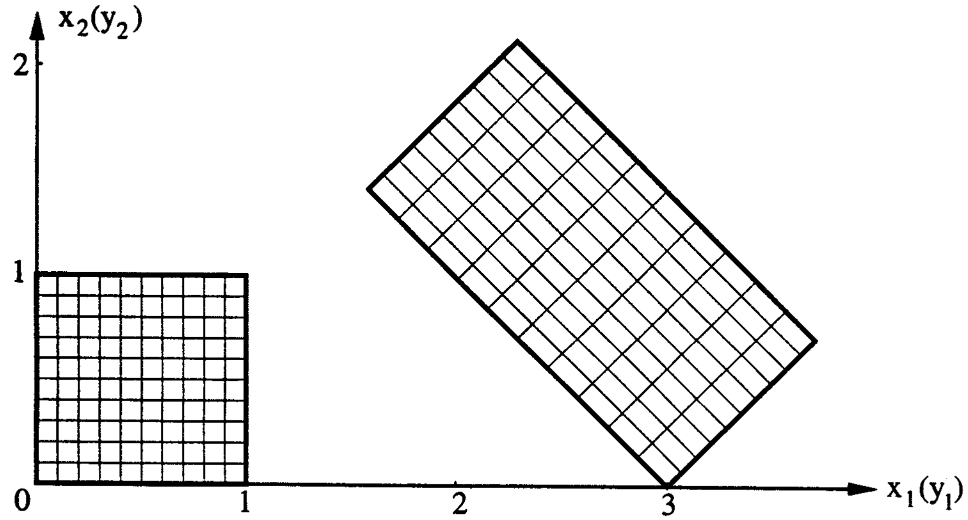


Figure 2: Example of a homogeneous deformation. The reference configuration on the left deforms to the deformed configuration on the right under the deformation described in Eq. (2.15).

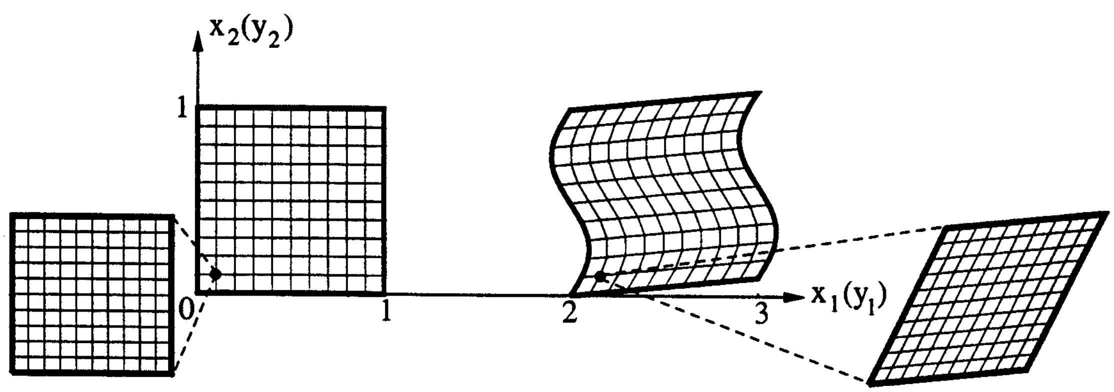


Figure 3: Example of an inhomogeneous deformation. The reference configuration on the left deforms to the deformed configuration on the right under the deformation described in Eq. (2.16). Notice that under sufficient magnification, an inhomogeneous deformation can be approximated by a homogeneous deformation.

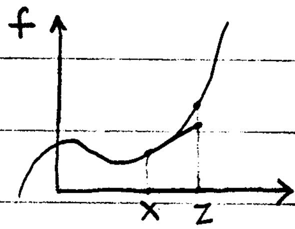
Consider the point $(\underset{\sim}{x} + d\underset{\sim}{x})$; it goes to $(\underset{\sim}{y} + d\underset{\sim}{y})$ after deformation.

$$d\underset{\sim}{x} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \qquad d\underset{\sim}{y} = \begin{pmatrix} dy_1 \\ dy_2 \\ dy_3 \end{pmatrix}$$

What is the relation between $d\underset{\sim}{x}$ and $d\underset{\sim}{y}$?

Recall the Taylor expansion of a function.

$$f(z) - f(x) = \frac{df}{dx}(x)\,(z - x) + O(|z-x|^2)$$



or $$df = \frac{df}{dx}(x)\,dx + O(|dx|^2)$$

Deformation gradient

$$\underset{\approx}{F} = \nabla \underset{\sim}{y} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{pmatrix}$$

$$F_{ij} = \frac{\partial y_i}{\partial x_j}$$

$$\boxed{d\underset{\sim}{y} = \underset{\approx}{F}(x)\, d\underset{\sim}{x}} \qquad \begin{pmatrix} dy_1 \\ dy_2 \\ dy_3 \end{pmatrix} = \begin{pmatrix} \qquad\qquad \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Example 1 Uniaxial deformation.

$$F = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2 Simple shear

$$F = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3 Pure bending

$$F = \begin{pmatrix} \frac{R+x_2}{R}\cos\left(\frac{x_1}{R}\right) & \sin\left(\frac{x_1}{R}\right) & 0 \\ -\frac{R+x_2}{R}\sin\left(\frac{x_1}{R}\right) & \cos\left(\frac{x_1}{R}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Infinitesimal Deformation

Recall the displacement $\underset{\sim}{u} = \underset{\sim}{y} - \underset{\sim}{x}$

$\therefore$ displacement gradient $= \nabla \underset{\approx}{u} = \nabla \underset{\approx}{y} - \underset{\approx}{I}$

A deformation is <u>infinitesimal</u> if $\nabla \underset{\approx}{u}$ is small

Example 1. Uniaxial deformation is infinitesimal if $\lambda - 1$ is small

Example 2. Simple shear is infinitesimal if if γ is small.

Example 3. Pure bending is infinitesimal if (L/R) is small.

Vector Algebra

Components of a vector form a column matrix $\Bigg\}$ $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Components of a tensor form a square matrix $\Bigg\}$ $\underset{\approx}{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

Tensors map vectors to vectors

$$\underset{\approx}{A}\,\underset{\sim}{a} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} A_{11}a_1 + A_{12}a_2 + \\ \\ \\ \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \underset{\sim}{b}$$

Indicial notation $\quad b_i = \sum_{j=1}^{3} A_{ij}\, a_j = A_{ij}\, a_j$

Summation convention - repeated incides are summed!

Dot product between two vectors:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$$

$$= (a_1 \ a_2 \ a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \underset{\sim}{a}^T \underset{\sim}{b}$$

Notice: $\underset{\sim}{b} \cdot (\underset{\approx}{A}\,\underset{\sim}{a}) = \underset{\sim}{b}^T \underset{\approx}{A}\,\underset{\sim}{a} = (\underset{\approx}{A}^T \underset{\sim}{b})^T \underset{\sim}{a}$

$$= \underset{\sim}{a} \cdot (\underset{\approx}{A}^T \underset{\sim}{b})$$

__Normal strain.__ Measures the stretch, where

$$\text{stretch} = \frac{\text{deformed length}}{\text{original length}}$$

The stretch suffered by $d\underset{\sim}{x} = \lambda = \dfrac{|d\underset{\sim}{y}|}{|d\underset{\sim}{x}|}$

$$\therefore \lambda = \frac{|d\underset{\sim}{y}|}{|d\underset{\sim}{x}|} = \frac{|\underset{\approx}{F}\, d\underset{\sim}{x}|}{|d\underset{\sim}{x}|} = \left| \underset{\approx}{F} \frac{d\underset{\sim}{x}}{|d\underset{\sim}{x}|} \right| = \left| \underset{\approx}{F}\, \hat{\underset{\sim}{e}} \right|$$

$\hat{e}$ is the unit vector in the direction of $d\underset{\sim}{x}$.

$$\therefore \lambda = |\underset{\approx}{F}\hat{\underset{\sim}{e}}| = \sqrt{(\underset{\approx}{F}\hat{\underset{\sim}{e}})\cdot(\underset{\approx}{F}\hat{\underset{\sim}{e}})} = \sqrt{\hat{\underset{\sim}{e}}\cdot(\underset{\approx}{F}^T\underset{\approx}{F}\,\hat{\underset{\sim}{e}})}$$

is the stretch in the direction $\hat{e}$.

$$\therefore \text{strain} = \frac{\text{change in length}}{\text{original length}} = \lambda - 1$$

$$= \sqrt{\hat{\underset{\sim}{e}}\cdot(\underset{\approx}{F}^T\underset{\approx}{F})\,\hat{\underset{\sim}{e}}} - 1 .$$

If the deformation is __infinitesimal__:

Recall : $\nabla\underset{\approx}{u} = \underset{\approx}{F} - \underset{\approx}{I}$ or $\underset{\approx}{F} = \nabla\underset{\approx}{u} + \underset{\approx}{I}$

$$\therefore \underset{\approx}{F}^T\underset{\approx}{F} = (\underset{\approx}{I} + \nabla\underset{\approx}{u}^T)(\underset{\approx}{I} + \nabla\underset{\approx}{u}) = \underset{\approx}{I} + \nabla\underset{\approx}{u} + \nabla\underset{\approx}{u}^T + \underbrace{\nabla\underset{\approx}{u}^T\nabla\underset{\approx}{u}}_{\text{small}}$$

or $\underset{\approx}{F}^T\underset{\approx}{F} \simeq \underset{\approx}{I} + \nabla\underset{\approx}{u} + \nabla\underset{\approx}{u}^T$

$\therefore$ $\left.\begin{array}{l}\text{strain in the}\\ \text{direction } \hat{e}\end{array}\right\} \simeq \sqrt{1 + \hat{\underset{\sim}{e}} \cdot \left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right) \hat{\underset{\sim}{e}}} - 1$

Recall : $(1+\varepsilon)^n \simeq 1 + n\varepsilon$

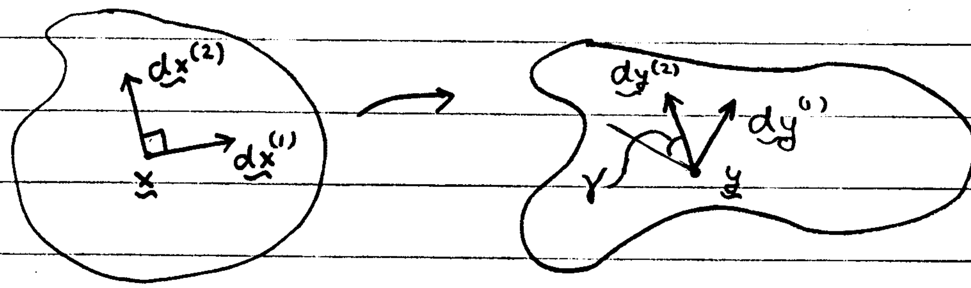
$\therefore$ $\left.\begin{array}{l}\text{normal strain in}\\ \text{the direction } \hat{e}\end{array}\right\} \simeq \hat{\underset{\sim}{e}} \cdot \left(\frac{1}{2}\left(\nabla \underset{\approx}{u} + \nabla \underset{\approx}{u}^T\right)\right) \hat{\underset{\sim}{e}}$

Infinitesimal strain tensor

$$\underset{\approx}{\varepsilon} = \frac{1}{2}\left(\nabla \underset{\approx}{u} + \nabla \underset{\approx}{u}^T\right)$$

$\left.\begin{array}{l}\text{Normal strain in}\\ \text{the direction } \hat{\underset{\sim}{e}}\end{array}\right\} = \hat{\underset{\sim}{e}} \cdot \underset{\approx}{\varepsilon}\, \underset{\sim}{e}$

Shear strain. Measures changes in angle



γ is the change in angle between two filaments which were mutually perpendicular.

$$\sin\gamma = \frac{d\underset{\sim}{y}^{(1)} \cdot d\underset{\sim}{y}^{(2)}}{|d\underset{\sim}{y}^{(1)}|\,|d\underset{\sim}{y}^{(2)}|}$$

$$|d\underset{\sim}{y}^{(1)}| = \lambda^{(1)}\,|d\underset{\sim}{x}^{(1)}|$$

$$|d\underset{\sim}{y}^{(2)}| = \lambda^{(2)}\,|d\underset{\sim}{x}^{(2)}|$$

$$= \frac{1}{\lambda^{(1)}\lambda^{(2)}} \; \frac{(\underset{\approx}{F}\,d\underset{\sim}{x}^{(1)})\cdot(\underset{\approx}{F}\,d\underset{\sim}{x}^{(2)})}{|d\underset{\sim}{x}^{(1)}|\,|d\underset{\sim}{x}^{(2)}|}$$

$$= \frac{1}{\lambda^{(1)}\lambda^{(2)}} \; \hat{\underset{\sim}{e}}_1 \cdot \underset{\approx}{F}^T \underset{\approx}{F}\, \hat{\underset{\sim}{e}}_2$$

$\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2$ are the unit vectors in the direction of $d\underset{\sim}{x}^{(1)}$, $d\underset{\sim}{x}^{(2)}$.

If the deformation is <u>infinitesimal</u>,

$$\sin\gamma \simeq \gamma \quad , \quad \lambda^{(1)} = 1 + \varepsilon^{(1)} \; ; \; \varepsilon^{(1)} = \hat{\underset{\sim}{e}}_1 \cdot \underset{\approx}{\varepsilon}\, \hat{\underset{\sim}{e}}_1$$

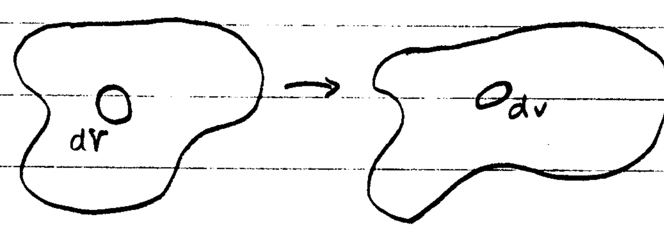
$$\lambda^{(2)} = 1 + \varepsilon^{(2)} \; ; \; \varepsilon^{(2)} = \hat{\underset{\sim}{e}}_2 \cdot \underset{\approx}{\varepsilon}\, \hat{\underset{\sim}{e}}_2$$

$$\therefore \left.\begin{array}{l}\text{the shear strain}\\ \text{between } \hat{e}_1 - \hat{e}_2\end{array}\right\} \simeq \gamma \simeq \frac{\hat{e}_1 \cdot \left(\underset{\approx}{I} + \nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right)\hat{e}_2}{\left(1+\varepsilon^{(1)}\right)\left(1+\varepsilon^{(2)}\right)}$$

$$\simeq \left[\underbrace{\hat{e}_1 \cdot \hat{e}_2}_{0} + \hat{\underset{\sim}{e}}_1 \cdot \left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right)\hat{e}_2\right]\left(1-\varepsilon^{(1)}\right)\left(1-\varepsilon^{(2)}\right)$$

$$\boxed{\therefore \left.\begin{array}{l}\text{Shear strain}\\ \text{between } \hat{e}_1 - \hat{e}_2\\ \text{directions}\end{array}\right\} = 2\, \hat{\underset{\sim}{e}}_1 \cdot \underset{\approx}{\varepsilon}\, \hat{\underset{\sim}{e}}_2}$$

<u>Volumetric strain</u>



$$dv = \left(\det \underset{\approx}{F}\right) dV$$

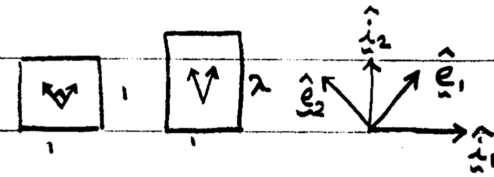
$$\therefore \text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \det \underset{\approx}{F} - 1$$

If the deformation is <u>infinitesimal</u>

$$\boxed{\therefore \text{volumetric strain} = \operatorname{trace} \underset{\approx}{\varepsilon} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}$$

Example 1. Uniaxial deformation.

$$\underset{\approx}{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$\varepsilon_{11} = \lambda - 1$

$$\hat{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \qquad \hat{e}_2 = \begin{pmatrix} -1/\sqrt{2} \\ +1/\sqrt{2} \\ 0 \end{pmatrix}$$

Normal strain:

$\hat{i}_1$ direction : ε_{11}

$\hat{i}_2$ direction : 0

$\hat{e}_1$ direction : $\frac{1}{2}\varepsilon_{11}$

$\hat{e}_2$ direction : $\frac{1}{2}\varepsilon_{11}$

$\varepsilon_{11} > 0 \cdots$ extension

$\varepsilon_{11} < 0 \cdots$ compression

Shear strain:

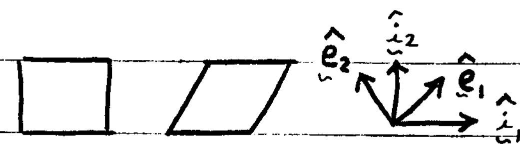
$\hat{i}_1 - \hat{i}_2$: 0

$$\hat{e}_1 - \hat{e}_2 : 2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon_{11}$$

Example 2

$$\underset{\sim}{\varepsilon} = \frac{1}{2}\begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Normal strain:

$\hat{\underset{\sim}{i}}_1$: 0

$\hat{\underset{\sim}{i}}_2$: 0

$\hat{\underset{\sim}{e}}_1$: $\frac{1}{2}\kappa$

$\hat{\underset{\sim}{e}}_2$: $-\frac{1}{2}\kappa$

$\left\{\begin{array}{l}\text{One diagonal becomes} \\ \text{longer while the other} \\ \text{becomes shorter!}\end{array}\right.$

Shear strain:

$\hat{\underset{\sim}{i}}_1 - \hat{\underset{\sim}{i}}_2$: κ

$\hat{\underset{\sim}{e}}_1 - \hat{\underset{\sim}{e}}_2$: 0

Notice in both examples that shear is associated with unequal stretch :

Results :

1. Infinitesimal strain tensor $\underset{\approx}{\varepsilon} = \frac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^{T}\right)$

2. $\left.\begin{array}{l}\text{Normal strain in} \\ \text{some direction } \hat{\underset{\sim}{e}}\end{array}\right\} = \hat{\underset{\sim}{e}} \cdot \underset{\approx}{\varepsilon}\, \hat{\underset{\sim}{e}}$

3. $\left.\begin{array}{l}\text{Shear strain in} \\ \text{the directions } \hat{e}_1 - \hat{e}_2\end{array}\right\} = 2\, \hat{\underset{\sim}{e}}_1 \cdot \underset{\approx}{\varepsilon}\, \hat{\underset{\sim}{e}}_2$

4. Volumetric strain = trace $\underset{\approx}{\varepsilon}$

5. To find the maximum strains and the directions in which they act:
   Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (numbered so that $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_3$) be the eigenvalues of $\underset{\approx}{\varepsilon}$ with corresponding mutually perpendicular eigenvectors $\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3$.

   (a) $\left.\begin{array}{l}\text{Largest tensile strain or} \\ \text{Smallest compressive strain}\end{array}\right\} = \varepsilon_3$ in the direction $\hat{\underset{\sim}{e}}_3$.

$$\max_{\varepsilon} \hat{e} \cdot \underset{\approx}{\varepsilon} \hat{e} = \max_{e} \left(e \cdot \varepsilon e - \lambda (e \cdot e - 1) \right)$$

Lagrange multiplier

$$\min_{g(x,y)=0} f(x,y) = \min_{(x,y)} f(x,y) - \lambda g(x,y)$$

$$\therefore \; 2 \underset{\approx}{\varepsilon} e - 2 \lambda e = 0 \quad \Rightarrow \quad \boxed{\underset{\approx}{\varepsilon} \hat{e} = \lambda \hat{e}}$$

$$\boxed{\therefore \max_{\hat{e}} \hat{e} \cdot \underset{\approx}{\varepsilon} \hat{e} = \text{max eigenvalue of } \underset{\approx}{\varepsilon}}$$

Spooky Man!

(b) Smallest tensile strain or largest compressive strain $\Big\} = \varepsilon_1$ in the direction $\hat{\underset{\sim}{e}}_1$.

(c) Largest shear strain $= \;(\varepsilon_3 - \varepsilon_1)$

in the directions

$$\tfrac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_3) - \tfrac{1}{\sqrt{2}}(\hat{e}_3 - \hat{e}_1)$$

6. The shear strain is zero in all directions if and only if

$\underset{\approx}{\varepsilon} = \alpha \underset{\approx}{I}$ ... purely volumetric strain

7. For any infinitesimal strain tensor,

$$\underset{\approx}{\varepsilon} = \underbrace{\left(\tfrac{1}{3}\operatorname{tr}\underset{\approx}{\varepsilon}\right)\underset{\approx}{I}}_{\text{Volumetric}} + \underbrace{\hat{\underset{\approx}{\varepsilon}}}_{\text{Deviatoric}}$$

$$\hat{\underset{\approx}{\varepsilon}} = \underset{\approx}{\varepsilon} - \left(\tfrac{1}{3}\operatorname{tr}\underset{\approx}{\varepsilon}\right)\underset{\approx}{I}.$$

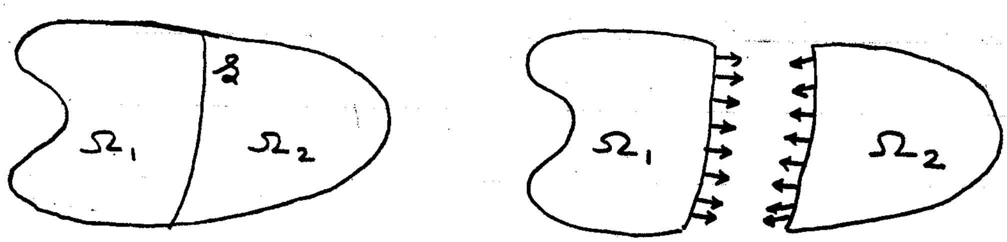
Notice that $\operatorname{tr}\hat{\underset{\approx}{\varepsilon}} = 0$ so that it has no volumetric strain.

Forces and Moments - Stress

We will confine this discussion to infinitesimal deformations and not distinguish between reference and deformed configuration. At the very end, we will state how to generalize this to finite deformations.

There are two types of forces that we consider

1. Contact forces: These forces act on surfaces in a body.



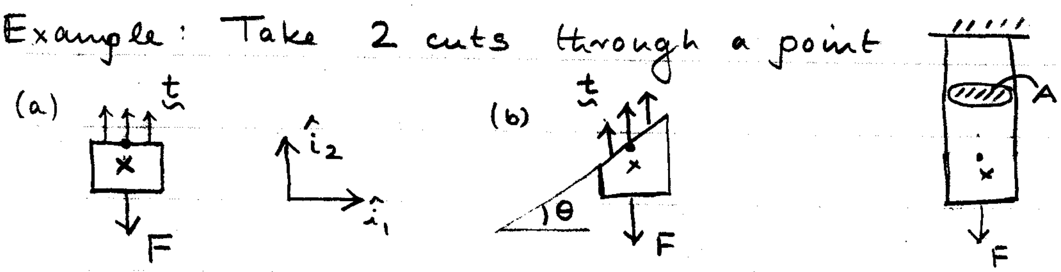
Suppose a surface $\mathcal{S}$ divides a body into two parts Ω_1 and Ω_2. Ω_1 knows of the presence of Ω_2 only through forces that act on the surface $\mathcal{S}$. Therefore we can replace Ω_2 with forces that act on $\mathcal{S}$. and Ω_1 would not know the difference.

We can describe these as a force per unit area $\mathbf{t}$. This is called the <u>traction</u> or the <u>stress vector</u>.

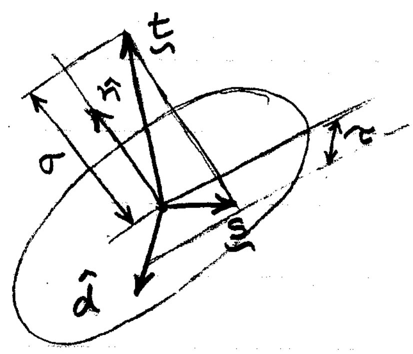
The stress vector can depend on both the position and also the normal to the surface

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, \hat{n}).$$

Example: Take 2 cuts through a point



$\mathbf{t} = \frac{F}{A}\hat{i}_2$ $\qquad\qquad$ $\mathbf{t} = \frac{F}{A}\cos\theta\,\hat{i}_2$



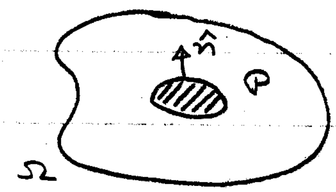
Consider a surface with normal $\hat{n}$. Let $\mathbf{t}$ be the traction.
We can decompose it as

$$\mathbf{t} = \sigma\hat{n} + \mathbf{s}$$

where $\mathbf{s}\cdot\hat{n} = 0$ so that $\mathbf{s}$ lies on the surface.
σ is called the <u>normal stress</u>
$\mathbf{s}$ is called the <u>shear stress</u>
Suppose τ is the component of $\mathbf{s}$ in some direction $\hat{d}$ on the surface. τ is the shear in the direction $\hat{d}$.

2. <u>Body forces:</u> These are like gravity and act on the entire body. We express them as force per unit volume $\underset{\sim}{b}$.

Let $\mathcal{P}$ be a part of the body Ω. Then,



(a) Sum of external forces on $\mathcal{P}$ $\Bigg\} = \displaystyle\int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{x})\, dV + \int_{\partial\mathcal{P}} \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\, dA$

(b) Sum of external moments on $\mathcal{P}$ about $\underset{\sim}{x}_o$ $\Bigg\} = \displaystyle\int_{\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_o) \times \underset{\sim}{b}(\underset{\sim}{x})\, dV + \int_{\partial\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_o) \times \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\, dA$

Note: 1. Above external means exteral to $\mathcal{P}$

2. Above $\hat{n}$ is the outward normal to $\partial\mathcal{P}$.

We now postulate two physical principles

1. Balance of forces: The sum of all external forces acting on any part $\mathcal{P}$ is zero:

$$\int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{x})\, dV + \int_{\partial\mathcal{P}} \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\, dA = 0$$

2. Balance of moments: The sum of all external moments (about any point $\underset{\sim}{x}_0$) acting on any part $\mathcal{P}$ is zero:

$$\int_{\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \underset{\sim}{b}(\underset{\sim}{x})\, dV + \int_{\partial\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\, dA = 0$$

Based on these 2 laws, we will show that

1. $\underset{\sim}{t}(\underset{\sim}{x}, \hat{n}) = \underset{\approx}{T}(\underset{\sim}{x})\, \hat{n}$ for a tensor $\underset{\approx}{T}$. which we will call the <u>stress tensor</u>

2. $\underset{\approx}{T}(\underset{\sim}{x}) = \underset{\approx}{T}^T(\underset{\sim}{x})$

3. $\mathrm{Div}\, \underset{\approx}{T} + \underset{\sim}{b} = 0$

i.e., $T_{ij,j} + b_i = 0$

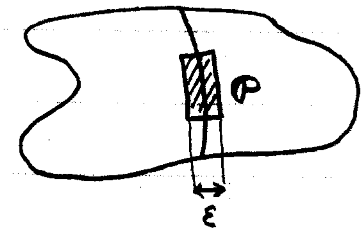
or $\dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{12}}{\partial x_2} + \dfrac{\partial T_{13}}{\partial x_3} + b_1 = 0$

$\dfrac{\partial T_{21}}{\partial x_1} + \ldots$

$\ldots$

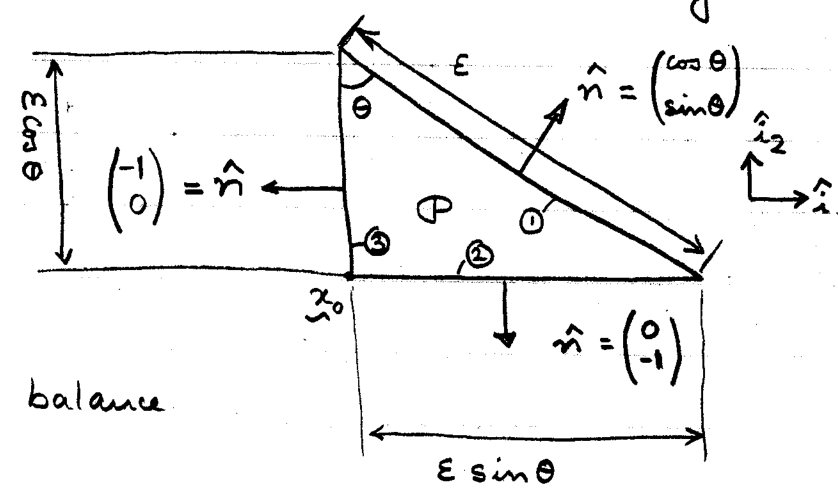
First observe $\underset{\sim}{t}(\underset{\sim}{x}, \hat{n}) = -\underset{\sim}{t}(\underset{\sim}{x}, -\hat{n})$

This is very easily seen by applying the principle of balance of forces on the part $\mathcal{P}$ shown and letting ε go to zero.



Let us now switch to 2 dimensions momentarily.

Consider any point $\underset{\sim}{x}_0$ in a body and the part $\mathcal{P}$ near it.



According to the balance of forces,

$$\int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{x})\,dV + \int_{①} \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\,dA + \int_{②} \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\,dA + \int_{③} \underset{\sim}{t}(\underset{\sim}{x}, \hat{n})\,dA = 0$$

If ε is small enough, we can replace x with x_0 above

$$\therefore \int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{x})\,dV \simeq \int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{x}_0)\,dV = \underset{\sim}{b}(\underset{\sim}{x}_0)\,\mathrm{vol}(\mathcal{P})$$

$$= \frac{\varepsilon^2}{2}\sin\theta\cos\theta\;\underset{\sim}{b}(\underset{\sim}{x}_0)$$

and so on.

$$\therefore \frac{\varepsilon^2}{2}\sin\theta\cos\theta\,\mathbf{b}(\mathbf{x}_0) + \varepsilon\,\mathbf{t}(\mathbf{x}_0,\hat{n}) + \varepsilon\sin\theta\;\mathbf{t}(\mathbf{x}_0,-\hat{\imath}_2)$$

$$+\,\varepsilon\cos\theta\;\mathbf{t}(x_0,-\hat{\imath}_1) = 0$$

Divide by ε and let $\varepsilon \to 0$.

$$\mathbf{t}(\mathbf{x}_0,\hat{n}) = \mathbf{t}(\mathbf{x}_0,\hat{\imath}_1)\cos\theta + \mathbf{t}(\mathbf{x}_0,\hat{\imath}_2)\sin\theta$$

$$\text{since}\begin{cases}\mathbf{t}(\mathbf{x}_0,-\hat{\imath}_1) = -\mathbf{t}(\mathbf{x}_0,\hat{\imath}_1)\\ \mathbf{t}(\mathbf{x}_0,-\hat{\imath}_2) = -\mathbf{t}(\mathbf{x}_0,\hat{\imath}_2)\end{cases}$$

Let $\mathbf{t}(\mathbf{x}_0,\hat{\imath}_1) = \mathbf{t}^{(1)}$, $\mathbf{t}(x_0,\hat{\imath}_2) = \mathbf{t}^{(2)}$

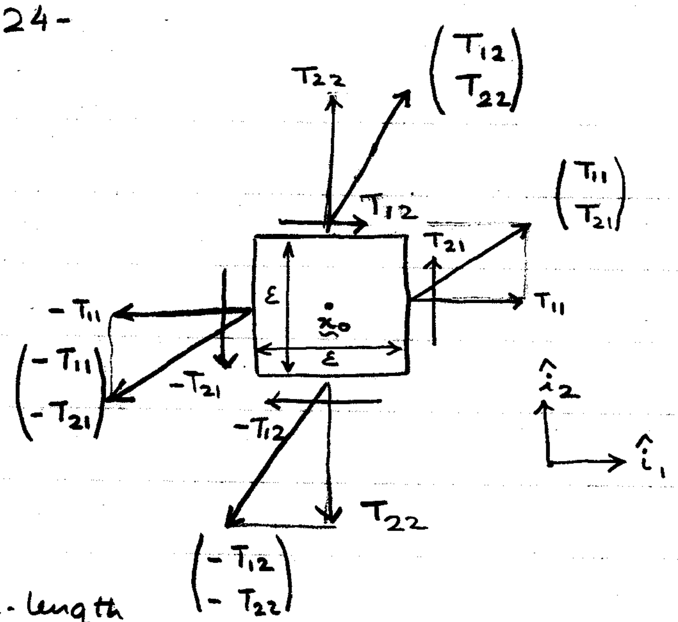
$$\therefore \mathbf{t}(\mathbf{x}_0,\hat{n}) = \begin{pmatrix} t_1^{(1)} & t_1^{(2)} \\ t_2^{(1)} & t_2^{(2)} \end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

Let $\mathbf{T}(\mathbf{x}_0)$ be the tensor with components shown above.

$$\therefore \boxed{\mathbf{t}(\mathbf{x}_0,\hat{n}) = \mathbf{T}(\mathbf{x}_0)\,\hat{n}}$$

$\mathbf{T}(x_0)$ is called the <u>stress tensor</u>.

We can use a similar argument in 3D, but with a tetrahedron



Let us use this to study the state of stress near a point $\underset{\sim}{x}_0$. Once again, confine ourself to 2D.

Consider a box of side-length ε around the point $\underset{\sim}{x}_0$. If ε is small, we can assume that the stress tensor is constant:

$$\underset{\approx}{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$\therefore$ the traction on the right face:

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$$

T_{11} is the normal stress acting on this face and T_{21} is the shear stress in the direction $\hat{i}_2$.

|||ly the traction on the left face

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -T_{11} \\ -T_{21} \end{pmatrix}$$

Ill^{ly} the traction on the top and bottom faces are given by

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \begin{pmatrix} \pm T_{12} \\ \pm T_{22} \end{pmatrix}.$$

Let us now use these to calculate the moments about $\underset{\sim}{x}_0$

$$\underbrace{\int_{P} (\underset{\sim}{x}-\underset{\sim}{x}_0) \times \underset{\sim}{b}(x)\, dV}_{O(\varepsilon^3)} + \underbrace{\int_{\partial P} (\underset{\sim}{x}-\underset{\sim}{x}_0) \times (\underset{\approx}{T}\, \hat{n})\, dA}_{\frac{\varepsilon^2}{2} T_{21} - \frac{\varepsilon^2}{2} T_{12} + \frac{\varepsilon^2}{2} T_{21} - \frac{\varepsilon^2}{2} T_{12}} = 0$$

$$\Rightarrow \quad T_{12} = T_{21}$$

Similarly in 3D, $T_{13} = T_{31}$ & $T_{23} = T_{31}$

$$\Rightarrow \quad \boxed{\underset{\approx}{T} = \underset{\approx}{T}^T}$$

Substituting this back into the balance of forces,

$$\int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{x})\, dV + \int_{\partial\mathcal{P}} \underset{\approx}{T}(\underset{\sim}{x})\, \hat{n}\, dA = 0$$

Recall the divergence theorem.

$$\int_{\partial\mathcal{P}} \underset{\sim}{v} \cdot \hat{n}\, dA = \int_{\mathcal{P}} \mathrm{Div}\, \underset{\sim}{v}\, dV$$ for any vector field $\underset{\sim}{v}(\underset{\sim}{x})$

or $$\int_{\partial\mathcal{P}} v_j \hat{n}_j\, dA = \int_{\mathcal{P}} \frac{\partial v_j}{\partial x_j}\, dV$$

Now set $v_j = T_{ij}$ for any fixed $i = 1, 2$ or 3

$$\therefore \int_{\partial\mathcal{P}} T_{ij} \hat{n}_j\, dA = \int_{\mathcal{P}} \frac{\partial T_{ij}}{\partial x_j}\, dV$$

Let $\mathrm{Div}\, \underset{\approx}{T}$ be the vector with components $\dfrac{\partial T_{ij}}{\partial x_j}$

$$\therefore \int_{\partial\mathcal{P}} \underset{\approx}{T}\, \hat{n}\, dA = \int_{\mathcal{P}} \mathrm{Div}\, \underset{\approx}{T}\, dV$$

$\therefore$ the balance of forces becomes

$$\int_{\mathcal{P}} (\mathrm{Div}\,\underset{\approx}{T} + \underset{\sim}{b})\, dV = 0$$

for all parts $\mathcal{P}$ in the body

$$\Rightarrow \boxed{\mathrm{Div}\,\underset{\approx}{T} + \underset{\sim}{b} = 0}$$ Equilibrium equation.

Analogy: Let $\int_a^b f(x)\,dx = 0 \quad \forall \; a, b$

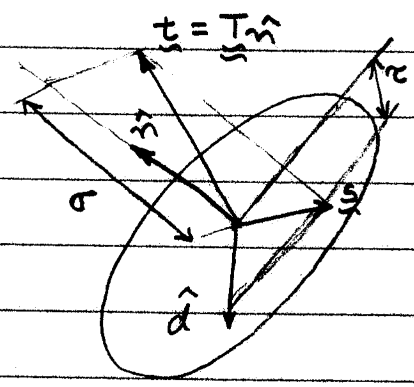
Then $f(x) = 0 \;\; \forall\, x$. Suppose not.

i.e., let $f(x_0) > 0$. Then $f(x) > 0 \;\; \forall\, x$ near x_0.

$\therefore$ if we choose a, b close enough to x_0,

$\int_a^b f(x)\,dx > 0$ contradicting the hypothesis

Let us now use the stress tensor to find the stresses on a surface. Consider the surface with normal $\hat{n}$ shown on page 19. The traction or the stress vector is given by



$$\underset{\sim}{t} = \underset{\approx}{T}\hat{n}$$

As before, we decompose it into normal and shear stress: $\underset{\sim}{t} = \underset{\approx}{T}\hat{n} = \sigma\hat{n} + \underset{\sim}{s}$

Normal stress : $\sigma = \underset{\sim}{t}\cdot\hat{n} = \hat{n}\cdot\underset{\approx}{T}\hat{n}$

$$\left.\begin{array}{l}\text{Shear stress in the}\\ \text{direction } \hat{d} \text{ (where}\\ \hat{d} \text{ is perpendicular}\\ \text{to } \hat{n})\end{array}\right\} \begin{array}{l}\tau = \underset{\sim}{s}\cdot\hat{d} = \underset{\sim}{t}\cdot\hat{d}\\ \quad = \hat{d}\cdot\underset{\approx}{T}\hat{n}\end{array}$$

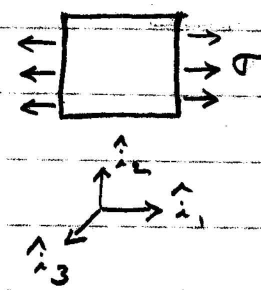
Recall that $\underset{\approx}{T} = \underset{\approx}{T}^T \Rightarrow \hat{d}\cdot\underset{\approx}{T}\hat{n} = \hat{n}\cdot\underset{\approx}{T}^T\hat{d} = \hat{n}\cdot\underset{\approx}{T}\hat{d}$

i.e., the shear stress in a direction $\hat{d}$ acting on a surface of normal $\hat{n}$ is equal to the shear stress in a direction $\hat{n}$ acting on a surface of normal $\hat{d}$.

$$\therefore \left.\begin{array}{l}\text{Normal stress in a}\\ \text{direction } \hat{n}\end{array}\right\} \sigma(\hat{n}) = \hat{n}\cdot\underset{\approx}{T}\hat{n}$$

$$\therefore \left.\begin{array}{l}\text{Shear stress in a pair}\\ \text{of mutually perpendicular}\\ \text{direction } \hat{n}, \hat{m}\end{array}\right\} \tau(\hat{n},\hat{m}) = \tau(\hat{m},\hat{n}) = \hat{n}\cdot\underset{\approx}{T}\hat{m}$$

Example 1. Uniaxial tension/compression



Consider an unit cube subjected to normal traction on two of its opposite faces as shown in the figure*. Find the stress tensor.

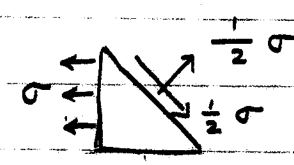
Clearly: $\hat{i}_1 \cdot \underset{\approx}{T}\, \hat{i}_1 = \sigma$ and

$\hat{i}_i \cdot \underset{\approx}{T}\, \hat{i}_j = 0$ if $\hat{i} \neq 1$, $\hat{j} \neq 1$

$$\therefore \quad \underset{\approx}{T} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consider a surface with normal $\hat{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. What are the stresses that act on it?

$$\underset{\sim}{t} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix}$$



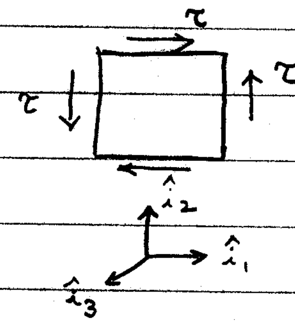
Normal stress $= t \cdot \hat{n} = \frac{1}{2}\sigma$

Shear stress in direction $\hat{\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$: $\frac{1}{2}\sigma$

* If σ is positive, we call it tension and if σ is negative we call it compression.

Example 2. Pure shear

Consider an unit cube subjected to the tractions shown. Find the stress tensor.



$$\left.\begin{aligned} \underset{\approx}{T}\,\hat{i}_1 &= \begin{pmatrix} 0 \\ \tau \\ 0 \end{pmatrix} \\ \underset{\approx}{T}\,\hat{i}_2 &= \begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix} \qquad \underset{\approx}{T}\,\hat{i}_3 = 0 \end{aligned}\right\} \Rightarrow \underset{\approx}{T} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

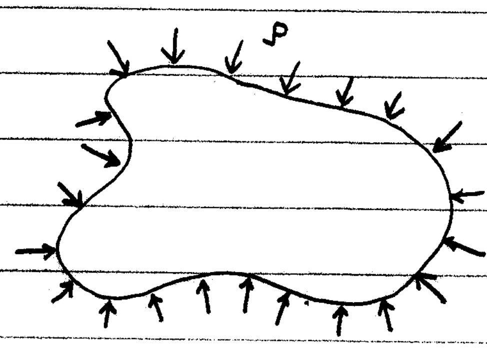
Consider a surface with normal $\hat{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. What are the stresses that act on it?

$$\underset{\sim}{t} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tau \\ \tau \\ 0 \end{pmatrix}$$

$\therefore$ Normal stress $\quad \sigma = \underset{\sim}{t}\cdot\hat{n} = \tau$

Shear stress $\quad = 0$

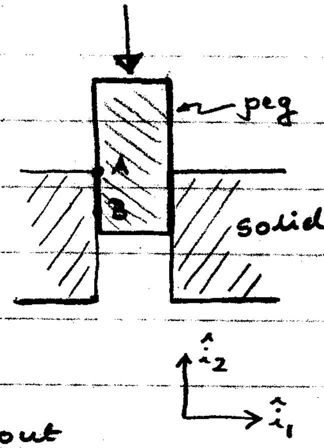
Example 3. Hydrostatic pressure



$\underset{\sim}{t} = -p\,\hat{n}$

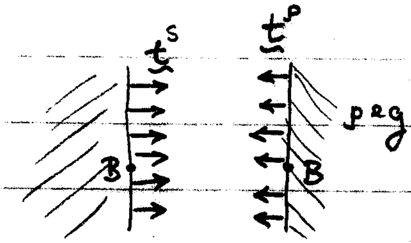
$$\therefore \underset{\approx}{T} = -p\,\underset{\approx}{I} = \begin{pmatrix} -p & & \\ & -p & \\ & & -p \end{pmatrix}$$

Show that the shear stress is zero on all surfaces



Example 4. Consider that we are forcing a peg into a very tight hole. What can you say about the stresses at points A and B? In particular what can you say about T_{11}, T_{22}, T_{12}? You can consider this as a 2D problem infinite in the third direction. (The answer remains unchanged for a circular peg in a circular hole!)

Notice first that the stress may be very different in the peg and the solid even close to the interface - the peg is compressed in the $\hat{i}_2$ direction, but the solid is not necessarily. Therefore, the stress tensor may jump across the interface! However, let us look close to the point B. Let $\underset{\sim}{t}^s$ be the traction acting on the solid and $\underset{\sim}{t}^p$ the traction acting on the peg at the point B.



Clearly, $\underset{\sim}{t}^s = \underset{\approx}{T}^s \hat{i}_1$, $\underset{\sim}{t}^p = \underset{\approx}{T}^p(-\hat{i}_1)$

However we are in equilibrium $\Rightarrow$ $\underset{\sim}{t}^s = -\underset{\approx}{t}^p$

$$\Rightarrow \left(\underset{\approx}{T}^{s} - \underset{\approx}{T}^{p}\right)\hat{i}_1 = 0$$

$\therefore$ at B : $\quad T_{11}^{p} = T_{11}^{s}$

$\qquad\qquad\qquad T_{12}^{p} = T_{12}^{s}$

III^{ly} at A : $\quad T_{11}^{p} = T_{11}^{s}, \quad T_{12}^{p} = T_{12}^{s}$

But we have more information at A.

Since the top surface of the solid is stress-free,

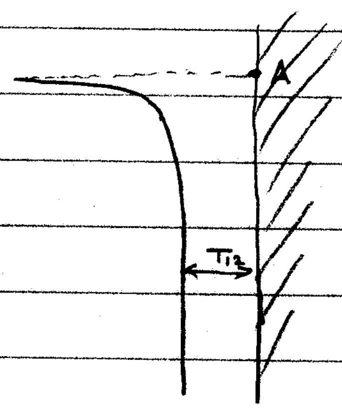
$$\underset{\approx}{T}^{s}\hat{i}_2 = 0 \quad \text{or} \quad T_{22}^{s} = T_{12}^{s} = 0$$

III^{ly} The lateral surface of the peg is stress-free,

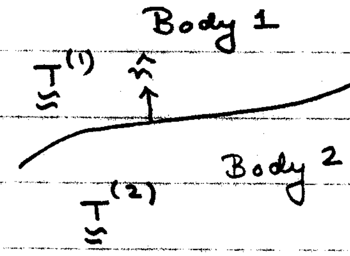
$$\therefore \quad -\underset{\approx}{T}^{p}\hat{i}_1 = 0 \quad \text{or} \quad T_{11}^{p} = T_{12}^{p} = 0$$

$\therefore$ at A : $\quad T_{11}^{p} = T_{11}^{s} = T_{12}^{p} = T_{12}^{s} = T_{22}^{s} = 0$!

Note: The shear T_{12} at the interface balances the applied force on the peg. Yet this zero at A. This results in a stress concentration at A. Infact elasticity theory predicts a singularity - the shear stress at various depths is as shown on the right.



<u>Traction continuity.</u>



We often encounter situations when two bodies are in very close contact or when two materials are bonded together as in a composite. In such situations the stress tensor can <u>jump</u> across the interface. Yet equilibrium requires $\underset{\sim}{t}(\underset{\sim}{x}, \hat{n}) = -\underset{\sim}{t}(\underset{\sim}{x}, -\hat{n})$

or $$\boxed{\left(\underset{\approx}{T}^{(1)} - \underset{\approx}{T}^{(2)}\right)\hat{n} = 0}$$

where $\underset{\approx}{T}^{(1)}$ and $\underset{\approx}{T}^{(2)}$ are the stress tensor on the two sides.

Summary and additional results.

1. We assume that there are two types of forces that act on a body
   1. Contact force $\underset{\sim}{t}(\underset{\sim}{x}, \hat{n})$
   2. Body force

2. We postulate two basic physical laws
   1. Balance of forces and 2. Balance of moments.

   This gives us
   (a) The stress tensor $\underset{\approx}{T}(x)$ which describes The contact forces
   $$\underset{\sim}{t}(\underset{\sim}{x}, \hat{n}) = \underset{\approx}{T}(x)\,\hat{n}.$$
   (b) $$\underset{\approx}{T} = \underset{\approx}{T}^T$$
   (c) $$\mathrm{Div}\,\underset{\approx}{T} + \underset{\sim}{b} = 0$$

3. To find the maximum stress and the directions in which they act:
   Let $\sigma_1, \sigma_2, \sigma_3$ (numbered so that $\sigma_1 \le \sigma_2 \le \sigma_3$) be the eigenvalues of $\underset{\approx}{T}$ with corresponding mutually perpendicular eigenvectors $\hat{s}_1, \hat{s}_2, \hat{s}_3$.

   (a) $\left.\begin{array}{l}\text{Largest tensile stress or}\\ \text{Smallest compressive stress}\end{array}\right\} = \sigma_3$ in the direction $\hat{s}_3$.

(b) Largest compressive stress or smallest tensile stress $\Big\} = \sigma_1$ in the direction $\hat{s}_1$.

(c) Largest shear stress $= \frac{1}{2}(\sigma_3 - \sigma_1)$ in the

directions : $\frac{1}{\sqrt{2}}(\hat{s}_1 + \hat{s}_3)$, $\frac{1}{\sqrt{2}}(\hat{s}_1 - \hat{s}_3)$

$\sigma_1, \sigma_2, \sigma_3$ ... principal stresses

$\hat{s}_1, \hat{s}_2, \hat{s}_3$ .... direction of principal stress.

Note that the shear stress is zero on a surface whose normal is a direction of principal stress.

4. The shear stress is zero in all direction if and only if $\underset{\approx}{T} = \alpha \underset{\approx}{I}$ ... hydrostatic pressure

5. Suppose $\operatorname{tr} \underset{\approx}{T} = 0$. Then, there is a coordinate system in which all the diagonal elements are zero. In other words, if we take a cube parallel to this coordinate system, the normal stress on all faces is zero.

6. Given any stress tensor,

$$\mathbf{T} = \underbrace{-p\,\mathbf{I}}_{\text{hydrostatic}} + \underbrace{\hat{\mathbf{T}}}_{\text{deviatoric}}$$

$$p = -\tfrac{1}{3}(\operatorname{tr}\mathbf{T})$$

$$\hat{\mathbf{T}} = \mathbf{T} + p\,\mathbf{I}$$

Notice $\operatorname{tr}\hat{\mathbf{T}} = 0$

7. If the net force on a part of the body is zero, and the net moment about a single point $\mathbf{x}_0$ is zero, then it follows that the net moment about any arbitrary point is zero.

Consider the moment about a point $\mathbf{z}$

$$\int_P (\mathbf{x}-\mathbf{z}) \times \mathbf{b}\, dV + \int_{\partial P} (\mathbf{x}-\mathbf{z}) \times \mathbf{t}(\mathbf{x},\hat{\mathbf{n}})\, dA$$

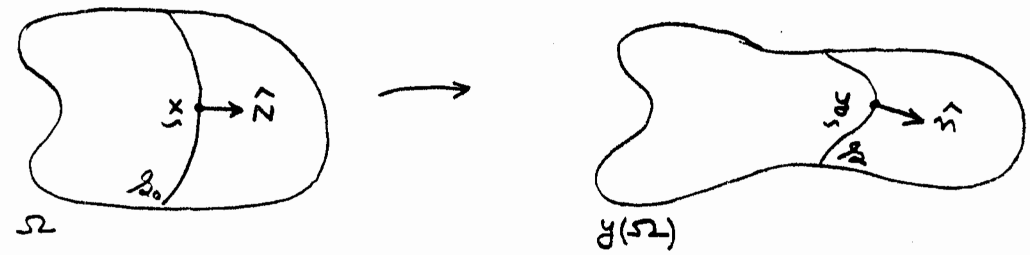
$$= \int_P (\mathbf{x}-\mathbf{x}_0+\mathbf{x}_0-\mathbf{z}) \times \mathbf{b}\, dV + \int_{\partial P} (\mathbf{x}-\mathbf{x}_0+\mathbf{x}_0-\mathbf{z}) \times \mathbf{t}\, dA$$

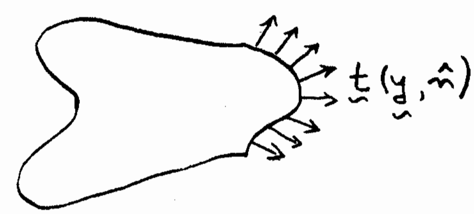
$$= \int_P (\mathbf{x}-\mathbf{x}_0) \times \mathbf{b}\, dV + \int_{\partial P} (\mathbf{x}-\mathbf{x}_0) \times \mathbf{t}\, dA$$

$$+ (\mathbf{x}_0-\mathbf{z}) \times \left[\int_P \mathbf{b}\, dV + \int_{\partial P} \mathbf{t}\, dA \right]$$

$$= 0.$$

Generalization of forces to finite deformations





Consider a body which occupies the region Ω in the reference configuration. After deformation, it goes to the region $y(\Omega)$. A material point $\underset{\sim}{x}$ in the reference configuration goes to $\underset{\sim}{y}$ in the deformed configuration. A surface $\mathcal{S}_0$ with normal $\hat{N}$ in the reference configuration goes to the surface $\mathcal{S}$ in the deformed conf.

There are 2 types of forces that act on the deformed body.

1. Contact forces: If we cut the deformed body along the surface $\mathcal{S}$, we can replace one part of The body with a traction or stress vector (force per unit deformed area)

$$\underset{\sim}{t}(\underset{\sim}{y}, \hat{n})$$

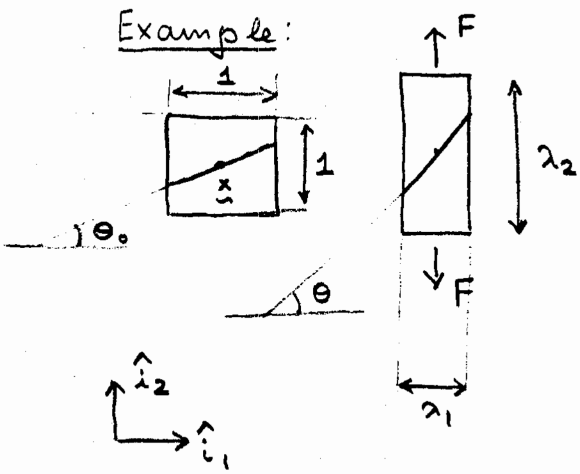
The traction depends on the point $\underset{\sim}{y}$ and on the normal to the surface.

Now, the point $\underset{\sim}{y}$ in the deformed configuration corresponds to the point $\underset{\sim}{x}$ in the reference configuration. Similarly the surface $\mathcal{S}$ and normal $\hat{n}$ correspond to the surface $\mathcal{S}_0$ and normal $\hat{N}$. Therefore, by making a change of variables, we can express the traction as a force per unit reference area

$$\underset{\sim}{S}(\underset{\sim}{x}, \hat{N}).$$

It is important to remember that the force acts on the deformed body; we are just expressing it as a function of reference quantities.

<u>Example</u>:



$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2$$
$$y_3 = x_3$$

A surface making an angle θ_0 goes to a surface making an angle θ. Elementary geometry shows

$$\tan\theta = \frac{\lambda_2}{\lambda_1} \operatorname{arctan} \theta_0 .$$

$$\hat{n} = \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} \qquad \hat{N} = \begin{pmatrix} -\cos\theta_0 \\ \sin\theta_0 \end{pmatrix}$$

$$\underset{\sim}{t}(\underset{\sim}{y}, \hat{n}) = \frac{F}{\lambda_1}\cos\theta \; \hat{i}_2 \qquad \underset{\sim}{S}(\underset{\sim}{x}, \hat{N}) = F\cos\theta_0 \; \hat{i}_2$$

2. Body force: Force per unit deformed volume

$$\underset{\sim}{b}(\underset{\sim}{y})$$

Once again, making a change of variables, we can express it as a force per unit reference volume

$$\underset{\sim}{B}(\underset{\sim}{x}).$$

Once again, the force acts in the deformed configuration; we are just expressing it as a function of references quantities.

Let us consider a part $\mathcal{P}_0$ in the reference configuration. After deformation it goes to $\mathcal{P}$.

Total external force acting on this part.

$$\int_{\partial\mathcal{P}} \underset{\sim}{t}(\underset{\sim}{y}, \hat{n})\, dA_y + \int_{\mathcal{P}} \underset{\sim}{b}(\underset{\sim}{y})\, dV_y$$

$$= \int_{\partial\mathcal{P}_0} \underset{\sim}{S}(\underset{\sim}{x}, \hat{N})\, dA_x + \int_{\mathcal{P}_0} \underset{\sim}{B}(\underset{\sim}{x})\, dV_x$$

dA_y ··· element of deformed area	dV_y ··· element of def. volume
dA_x ··· element of reference area	dV_x ··· element of ref. volume

Total external moment about some point $\underset{\sim}{c}$ on this part

$$\int_{\partial P} (\underset{\sim}{y} - \underset{\sim}{c}) \times \underset{\sim}{t}(\underset{\sim}{y}, \hat{n})\, dA_y + \int_{P} (\underset{\sim}{y} - \underset{\sim}{c}) \times \underset{\sim}{b}(\underset{\sim}{y})\, dV_y$$

$$= \int_{\partial P_0} (\underset{\sim}{y}(\underset{\sim}{x}) - \underset{\sim}{c}) \times \underset{\sim}{s}(\underset{\sim}{x}, \hat{N})\, dA_x + \int_{P_0} (\underset{\sim}{y}(\underset{\sim}{x}) - \underset{\sim}{c}) \times \underset{\sim}{B}(\underset{\sim}{x})\, dV_x$$

$\therefore$ we can write the total forces and moments in terms of either the reference or the deformed quantities.

Principle of balance of forces.
The total external force acting on any part of the body is zero.
Principle of balance of moments.
The total external moment acting on any part of the body is zero.

We can write this in terms of either the reference or the deformed quantities.

Based on these principles, we have the following results.

1. $\underset{\sim}{t}(\underset{\sim}{y}, \hat{n}) = -\underset{\sim}{t}(\underset{\sim}{y}, -\hat{n})$; $\underset{\sim}{s}(\underset{\sim}{x}, \hat{N}) = -\underset{\sim}{s}(\underset{\sim}{x}, -\hat{N})$

2. $\underset{\sim}{t}(\underset{\sim}{y}, \hat{n}) = \underset{\approx}{T}(\underset{\sim}{y})\, \hat{n}$; $\underset{\sim}{s}(\underset{\sim}{x}, \hat{N}) = \underset{\approx}{S}(\underset{\sim}{x})\, \hat{N}$

$\underset{\approx}{T}$.... <u>Cauchy stress</u>. It describes the force per unit deformed area. Also known as <u>true stress</u>.

$\underset{\approx}{S}$.... <u>Piola-Kirchhoff</u> or <u>nominal stress</u>. It describes the force per unit reference area. Also known as <u>engineering stress</u>.

It is possible to show that

$$\underset{\approx}{S}(\underset{\sim}{x}) = \left(\det \underset{\approx}{F}(\underset{\sim}{x})\right) \underset{\approx}{F}^{-1}(\underset{\sim}{x})\, \underset{\approx}{T}\left(\underset{\sim}{y}(\underset{\sim}{x})\right)$$

$$\text{or } \underset{\approx}{S} = (\det \underset{\approx}{F})\, \underset{\approx}{F}^{-1} \underset{\approx}{T}$$

where $F = \nabla y$ is the deformation gradient.

3. $\underset{\approx}{T} = \underset{\approx}{T}^{T}$; $\underset{\approx}{F}\underset{\approx}{S} = \underset{\approx}{S}^{T}\underset{\approx}{F}^{T} = (\underset{\approx}{F}\underset{\approx}{S})^{T}$

Note that the Cauchy stress is symmetric, but the Piola-Kirchhoff is not necessarily symmetric

4. $\operatorname{div} \underset{\sim}{\underset{\sim}{T}} + \underset{\sim}{b} = 0 \qquad (\operatorname{div} \underset{\sim}{\underset{\sim}{T}})_i = \dfrac{\partial T_{ij}}{\partial y_j}$

$\operatorname{Div} \underset{\sim}{\underset{\sim}{S}} + \underset{\sim}{B} = 0 \qquad (\operatorname{Div} \underset{\sim}{\underset{\sim}{S}})_i = \dfrac{\partial S_{ij}}{\partial x_j}$

In summary, under finite deformations, one can choose to express the forces in terms of reference quantities or deformed quantities. Depending on the choice one gets different stress tensors, However the basic principles are still the same.

Generalization of kinematics and forces to bodies in motion

Though the body is in motion, we will assume that the deformation is infinitesimal.

Deformation ... $\underset{\sim}{y}(\underset{\sim}{x}, t)$

Displacement .... $\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{y}(\underset{\sim}{x}, t) - \underset{\sim}{x}$

Particle velocity ... $\underset{\sim}{v}(\underset{\sim}{x}, t) = \dfrac{\partial \underset{\sim}{y}}{\partial t} = \dfrac{\partial \underset{\sim}{u}}{\partial t}$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y_1}{\partial t} \\ \dfrac{\partial y_2}{\partial t} \\ \dfrac{\partial y_3}{\partial t} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u_1}{\partial t} \\ \dfrac{\partial u_2}{\partial t} \\ \dfrac{\partial u_3}{\partial t} \end{pmatrix}$$

Particle acceleration ··· $\underset{\sim}{a}(\underset{\sim}{x}, t) = \dfrac{\partial \underset{\sim}{v}}{\partial t} = \dfrac{\partial^2 \underset{\sim}{y}}{\partial t^2} = \dfrac{\partial^2 \underset{\sim}{u}}{\partial t^2}$

Velocity gradient $\nabla \underset{\sim}{v} = \dfrac{\partial v_i}{\partial x_j} = \begin{pmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial x_2} & \cdot \\ \dfrac{\partial v_2}{\partial x_1} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

Rate of deformation $\underset{\approx}{D} = \dfrac{1}{2}\left(\nabla \underset{\sim}{v} + \nabla \underset{\sim}{v}^T\right) = \dfrac{\partial}{\partial t} \underset{\approx}{\varepsilon}$.

Once again, we assume two types of forces act on the body - contact forces and body forces.

Linear momentum in a part $\mathcal{P}$ of the body

$$\int_{\mathcal{P}} \rho \underset{\sim}{v} \, dV \qquad \rho \ldots \text{density}$$

Angular moment about x_0 in a part $\mathcal{P}$ of the body

$$\int_{\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_0) \times (\rho \underset{\sim}{v}) \, dV$$

Principle of balance of linear momentum
The total external force acting on a part $\mathcal{P}$ of a body is equal to the rate of change of linear momentum

$$\int_{\mathcal{P}} \underset{\sim}{b} \, dV + \int_{\partial \mathcal{P}} \underset{\sim}{t} \, dA = \frac{d}{dt} \int_{\mathcal{P}} \rho \underset{\sim}{v} \, dV$$

Since the deformation is infinitesimal, the density remains unchanged. Therefore

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \underset{\sim}{v}\, dV = \int_{\mathcal{P}} \rho \underset{\sim}{a}\, dV$$

$$\therefore \int_{\mathcal{P}} \underset{\sim}{b}\, dV + \int_{\partial\mathcal{P}} \underset{\sim}{t}\, dA = \int_{\mathcal{P}} \rho \underset{\sim}{a}\, dV$$

Principle of balance of angular momentum.
The total external moment acting on a part $\mathcal{P}$ of a body is equal to the rate of change of angular momentum.

$$\int_{\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_o) \times \underset{\sim}{b}\, dV + \int_{\partial\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_o) \times \underset{\sim}{t}\, dA = \frac{d}{dt}\int_{\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_o) \times (\rho \underset{\sim}{v})\, dV$$

$$= \int_{\mathcal{P}} (\underset{\sim}{x} - \underset{\sim}{x}_o) \times (\rho \underset{\sim}{a})\, dV$$

Based on these principles, we can show

1. $\underset{\sim}{t}(\underset{\sim}{x}, \hat{n}) = \underset{\approx}{T}(\underset{\sim}{x})\, \hat{n}$

2. $\underset{\approx}{T} = \underset{\approx}{T}^{T}$

3. $\operatorname{Div} \underset{\approx}{T} + \underset{\sim}{b} = \rho \underset{\sim}{a}$