

AE/AM/CE/ME 102b Homework 2 Solution

Mechanics of Structures and Solids - Winter 2012

Problem 1

Consider a solid neo-Hookean cylinder of undeformed length L and radius a subjected to a twisting moment M .

a) Find the relation between M and the angle of twist τ per unit length.

Hint: Assume a deformation of the form

$$r = f(R) \tag{1}$$

$$\theta = \hat{\theta} + \tau\lambda Z \tag{2}$$

$$z = \lambda Z \tag{3}$$

in cylindrical coordinates for λ constant.

Solution

Two ways to calculate the deformation gradient \underline{F} in cylindrical coordinates:

Method 1 Working from the definition of \underline{F} that \underline{F} transforms a relative position vector \underline{dx} between two close points in the reference configuration to the relative position vector between those two points in the deformed configuration:

$$\underline{dy} = \underline{F} \cdot \underline{dx} \tag{4}$$

Method 2 Using the vector formula form

$$\underline{F} = \underline{y} \otimes \nabla_{\underline{x}}, \tag{5}$$

Keep in mind that the physical components of \underline{dx} , \underline{dy} , \underline{y} and $\nabla_{\underline{x}}$ are used here to calculate the physical components of \underline{F} .

Notation:

- $\hat{i}_1, \hat{i}_2, \hat{i}_3$ are basis vectors of the Cartesian coordinates.
- $\{x_1, x_2, x_3\} = \{R, \hat{\theta}, Z\}$ are the cylindrical coordinates in the reference configuration. The subscripts 1, 2, 3 and $R, \hat{\theta}, Z$ will be used interchangeably.

- $\{\underline{e}_1^0, \underline{e}_2^0, \underline{e}_3^0\} = \{\underline{e}_R^0, \underline{e}_\theta^0, \underline{e}_Z^0\}$ are unit basis vectors of the cylindrical coordinates in the reference configuration, $\underline{e}_R^0 = \cos \hat{\theta} \underline{i}_1 + \sin \hat{\theta} \underline{i}_2, \underline{e}_\theta^0 = -\sin \hat{\theta} \underline{i}_1 + \cos \hat{\theta} \underline{i}_2, \underline{e}_Z^0 = \underline{i}_3$.
- $\{\eta_1, \eta_2, \eta_3\}$: reference configuration Cartesian coordinates, $\eta_1 = R \cos \hat{\theta}, \eta_2 = R \sin \hat{\theta}, \eta_3 = Z$.
- $\{y_1, y_2, y_3\} = \{r, \theta, z\}$: deformed cylindrical coordinates.
- $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\underline{e}_r, \underline{e}_\theta, \underline{e}_z\}$: unit basis vectors of the cylindrical coordinates in the deformed configuration, $\underline{e}_r = \cos \theta \underline{i}_1 + \sin \theta \underline{i}_2, \underline{e}_\theta = -\sin \theta \underline{i}_1 + \cos \theta \underline{i}_2, \underline{e}_z = \underline{i}_3$.
- $\{\xi_1, \xi_2, \xi_3\}$; deformed Cartesian coordinates, $\xi_1 = r \cos \theta, \xi_2 = r \sin \theta, \xi_3 = z$.

Note that r, θ, z are the cylindrical coordinate of the position vector \underline{y} , but they are **not** the physical component of \underline{y} , i.e. $\underline{y} \neq r \underline{e}_r + \theta \underline{e}_\theta + z \underline{e}_z = y_i \underline{e}_i$. Similarly, $dr, d\theta, dz$ are not the physical components of \underline{dy} , i.e. $\underline{dy} \neq dr \underline{e}_r + d\theta \underline{e}_\theta + dz \underline{e}_z$. To clarify about the physical component idea, let's calculate the physical component of \underline{y} and \underline{dy} .

$$\underline{y} = \xi_i \underline{i}_i = r \cos \theta \underline{i}_1 + r \sin \theta \underline{i}_2 + z \underline{i}_3 = r \underline{e}_r + z \underline{e}_z = y_{\langle i \rangle} \underline{e}_i$$

where the bracket subscription, $\langle . \rangle$, denotes the physical component. Therefore,

$$y_{\langle 1 \rangle} = r, y_{\langle 2 \rangle} = 0, y_{\langle 3 \rangle} = 0. \quad (6)$$

From the length ds between two close points:

$$\begin{aligned} ds^2 &= d\xi_i d\xi_i \equiv d\xi \cdot d\xi \\ &= dr^2 + r^2 d\theta^2 + dz^2 \equiv \underline{dy} \cdot \underline{dy} \end{aligned}$$

where $\underline{d\xi} = \xi_j \underline{i}_j$ and

$$\begin{aligned} \underline{dy} &= 1 \cdot dr \cdot \underline{e}_r + r \cdot d\theta \cdot \underline{e}_\theta + 1 \cdot dz \cdot \underline{e}_z \\ &= \sum_k h_k dy_k \underline{e}_k \\ &= dy_{\langle k \rangle} \underline{e}_k. \end{aligned}$$

$h_1 = 1, h_2 = r, h_3 = 1$ are called scaling factors for cylindrical coordinates;

$$dy_{\langle k \rangle} = h_k dy_k \quad (\text{no summation on } k) \quad (7)$$

are physical component of vector \underline{dy} . Similarly, for the reference configuration we have $h_1^0 = 1, h_2^0 = R, h_3^0 = 1$ and $\underline{dx}_{\langle k \rangle} = h_k^0 dx_k$ (no summation on k).

Now using the above two methods to calculate \underline{F} .

Method 1 By definition:

$$\underline{dy} = dy_{\langle k \rangle} \underline{e}_k, \quad \underline{dx} = dx_{\langle k \rangle} \underline{e}_k^0, \quad \underline{F} = F_{\langle km \rangle} \underline{e}_k \otimes \underline{e}_m^0 \quad (8)$$

and from equation (4)

$$\underline{dy} = \underline{F} \cdot \underline{dx} \quad \Rightarrow \quad dy_{\langle k \rangle} = F_{\langle km \rangle} dx_{\langle m \rangle} \quad (9)$$

Using the the calculation of physical quantities above ($dx_{\langle k \rangle} = h_k^0 dx_k$, $dy_{\langle k \rangle} = h_k dy_k$, no summation on k) and using the chain rule we have

$$dy_{\langle k \rangle} = h_k dy_k = h_k \frac{\partial y_k}{\partial x_m} dx_m = \Sigma_m \frac{h_k}{h_m^0} \frac{\partial y_k}{\partial x_m} dx_{\langle m \rangle} \quad (10)$$

$$(9) \text{ and } (10) \Rightarrow F_{\langle km \rangle} = \frac{h_k}{h_m^0} \frac{\partial y_k}{\partial x_m} \quad (\text{no summation on } m) \quad (11)$$

or

$$\underline{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \hat{\theta}} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial R} & \frac{r}{R} \frac{\partial \theta}{\partial \hat{\theta}} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \hat{\theta}} & \frac{\partial z}{\partial Z} \end{pmatrix} \quad (12)$$

Method 2 We have $\nabla_{\underline{x}}$ expressed in its physical component as:

$$\nabla_{\underline{x}} = \underline{e}_R^0 \frac{\partial}{\partial R} + \underline{e}_{\hat{\theta}}^0 + \frac{1}{R} \frac{\partial}{\partial \hat{\theta}} + \underline{e}_Z^0 \frac{\partial}{\partial Z}$$

and note that

$$\frac{\partial \underline{e}_r}{\partial R} = \frac{\partial \theta}{\partial R} \underline{e}_{\theta}, \quad \frac{\partial \underline{e}_r}{\partial \hat{\theta}} = \frac{\partial \theta}{\partial \hat{\theta}} \underline{e}_{\theta}, \quad \frac{\partial \underline{e}_r}{\partial Z} = \frac{\partial \theta}{\partial Z} \underline{e}_{\theta}, \quad \frac{\partial \underline{e}_z}{\partial x_i} = 0.$$

The first column of \underline{F} (associated with \underline{e}_R^0):

$$\begin{aligned} F_{\langle iR \rangle} \underline{e}_i \otimes \underline{e}_R^0 &= (y_{\langle i \rangle} \underline{e}_i) \otimes \underline{e}_R^0 \frac{\partial}{\partial R} = \frac{\partial}{\partial R} (r \underline{e}_r + z \underline{e}_z) \otimes \underline{e}_R^0 \\ &= \frac{\partial r}{\partial R} \underline{e}_r \otimes \underline{e}_R^0 + r \frac{\partial \theta}{\partial R} \underline{e}_{\theta} \otimes \underline{e}_R^0 + \frac{\partial z}{\partial R} \underline{e}_z \otimes \underline{e}_R^0 \end{aligned} \quad (13)$$

The second column of \underline{F} (associated with $\underline{e}_{\hat{\theta}}^0$):

$$\begin{aligned} F_{\langle i\hat{\theta} \rangle} \underline{e}_i \otimes \underline{e}_{\hat{\theta}}^0 &= (y_{\langle i \rangle} \underline{e}_i) \otimes \underline{e}_{\hat{\theta}}^0 \frac{1}{R} \frac{\partial}{\partial \hat{\theta}} = \frac{1}{R} \frac{\partial}{\partial \hat{\theta}} (r \underline{e}_r + z \underline{e}_z) \otimes \underline{e}_{\hat{\theta}}^0 \\ &= \frac{1}{R} \frac{\partial r}{\partial \hat{\theta}} \underline{e}_r \otimes \underline{e}_{\hat{\theta}}^0 + \frac{r}{R} \frac{\partial \theta}{\partial \hat{\theta}} \underline{e}_{\theta} \otimes \underline{e}_{\hat{\theta}}^0 + \frac{1}{R} \frac{\partial z}{\partial \hat{\theta}} \underline{e}_z \otimes \underline{e}_{\hat{\theta}}^0 \end{aligned} \quad (14)$$

The third column of \underline{F} (associated with \underline{e}_Z^0):

$$\begin{aligned} F_{\langle iZ \rangle} \underline{e}_i \otimes \underline{e}_Z^0 &= (y_{\langle i \rangle} \underline{e}_i) \otimes \underline{e}_Z^0 \frac{\partial}{\partial Z} = \frac{\partial}{\partial Z} (r \underline{e}_r + z \underline{e}_z) \otimes \underline{e}_Z^0 \\ &= \frac{\partial r}{\partial Z} \underline{e}_r \otimes \underline{e}_Z^0 + r \frac{\partial \theta}{\partial Z} \underline{e}_{\theta} \otimes \underline{e}_Z^0 + \frac{\partial z}{\partial Z} \underline{e}_z \otimes \underline{e}_Z^0 \end{aligned} \quad (15)$$

$$\Rightarrow \underline{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \hat{\theta}} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial R} & \frac{r}{R} \frac{\partial \theta}{\partial \hat{\theta}} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \hat{\theta}} & \frac{\partial z}{\partial Z} \end{pmatrix} \quad (16)$$

We can double check from (12) and (16) that both methods gives the same deformation gradient result.

Assuming our deformation of the form $r = f(R), \theta = \hat{\theta} + \tau\lambda Z, z = \lambda Z$, we have

$$\underline{F} = \begin{pmatrix} f'(R) & 0 & 0 \\ 0 & f(R)/R & f(R)\tau\lambda \\ 0 & 0 & \lambda \end{pmatrix} \quad (17)$$

Incompressibility:

$$\begin{aligned} \det(\underline{F}) = 1 &\Rightarrow \frac{f'(R)f(R)\lambda}{R} = 1 \Rightarrow fdf = \frac{R}{\lambda}dR \Rightarrow f^2(R) = \frac{R^2}{\lambda} + C \\ \Rightarrow r = f(R) &= \sqrt{\frac{R^2}{\lambda} + C} = \frac{R}{\sqrt{\lambda}} \quad (C = 0 - \text{no hole in the center}) \end{aligned} \quad (18)$$

$$\begin{aligned} \Rightarrow \underline{B} = \underline{F}\underline{F}^T &= \begin{pmatrix} f'(R)^2 & 0 & 0 \\ 0 & f(R)^2/R^2 + \lambda^2\tau^2 f(R)^2 & f(R)\tau\lambda^2 \\ 0 & f(R)\tau\lambda^2 & \lambda^2 \end{pmatrix} \\ &= \begin{pmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda + r^2\lambda^2\tau^2 & r\lambda^2\tau \\ 0 & r\lambda^2\tau & \lambda^2 \end{pmatrix} \end{aligned} \quad (19)$$

For a neo-Hookean material: $\underline{T} = -p\underline{I} + 2\alpha\underline{B}$

$$\Rightarrow \underline{T} = \begin{pmatrix} -p + 2\alpha/\lambda & 0 & 0 \\ 0 & -p + 2\alpha(1/\lambda + r^2\lambda^2\tau^2) & 2\alpha r\lambda^2\tau \\ 0 & 2\alpha r\lambda^2\tau & -p + 2\alpha\lambda^2 \end{pmatrix} \quad (20)$$

Total twisting moment in terms of the angle of twist:

$$M = \int_0^{r(a)} r T_{(\theta z)} 2\pi r dr = 4\pi\alpha\lambda^2\tau \int_0^{a/\sqrt{\lambda}} r^3 dr = \pi\alpha\lambda^2\tau r^4 \Big|_0^{a/\sqrt{\lambda}} = \pi\alpha a^4\tau \quad (21)$$

b) Will the radius of the cylinder change?

Solution: From part (a) we have $r = \frac{R}{\sqrt{\lambda}}$, so to answer this question we will find λ from the equilibrium equations. Note that p is **not** a constant but a function of r, θ , and z .

Equilibrium equations, $\nabla \cdot \underline{T} = 0$, in cylindrical coordinates, this case reduces to

$$\begin{aligned} \frac{\partial T_{\langle rr \rangle}}{\partial r} + \frac{T_{\langle rr \rangle} - T_{\langle \theta \theta \rangle}}{r} &= 0, \\ \frac{\partial T_{\langle \theta \theta \rangle}}{\partial \theta} = 0 &\Rightarrow \frac{\partial p}{\partial \theta} = 0, \\ \frac{\partial T_{\langle zz \rangle}}{\partial z} = 0 &\Rightarrow \frac{\partial p}{\partial z} = 0 \end{aligned} \quad (22)$$

so $p = p(r)$ and the first equilibrium equation becomes

$$\frac{dp}{dr} - 2\alpha\lambda^2\tau^2r = 0 \Rightarrow p(r) = -\alpha\lambda^2\tau^2r^2 + p_0 \quad (23)$$

Traction free at the side: $\left(\underline{T} \Big|_{r=a/\sqrt{\lambda}}\right) \cdot \underline{n} = 0$, where $\underline{n} = 1.\underline{e}_r + 0.\underline{e}_\theta + 0.\underline{e}_z$

$$\begin{aligned} \Rightarrow p(a/\sqrt{\lambda}) &= 2\alpha/\lambda & \Rightarrow p_0 &= 2\alpha/\lambda + \alpha\lambda\tau^2a^2 \\ & & \Rightarrow p &= \alpha\lambda\tau^2(a^2 - \lambda r^2) + 2\alpha/\lambda \end{aligned} \quad (24)$$

Force free at the top and bottom surfaces:

$$\int_A \left(\underline{T} \Big|_{z=0, z(L)}\right) \cdot \underline{e}_z dA = 0 \quad (25)$$

so the force acting on those surfaces in z direction can be calculated as

$$f_z = \int_A \underline{e}_z \cdot \underline{T} \cdot \underline{e}_z dA = 0 \quad (26)$$

$$f_z = \int_0^{a/\sqrt{\lambda}} 2\pi r \{-\alpha\lambda\tau^2(a^2 - \lambda r^2) - 2\alpha/\lambda + 2\alpha\lambda^2\} dr = \frac{a^2\pi\alpha(a^2\lambda^2\tau^2 - 4\lambda^3 + 4)}{2\lambda^2} \quad (27)$$

$$f_z \Big|_{\text{at } \lambda=1} = -\frac{1}{2}a^4\alpha\tau^2 \neq 0 \quad (28)$$

Therefore, $\lambda = 1$ is not a solution of the problem and the radius $r = R/\sqrt{\lambda}$ will change.

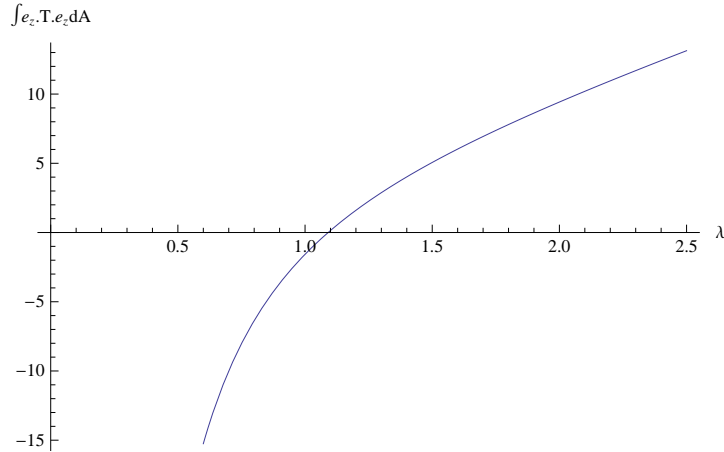


Figure 1: Example solution from the condition $f_z = 0$ with $a = 1, \tau = 1, \alpha = 1$ (notice that $f_z = 0$ at $\lambda \neq 1$)

c) Will the length of the cylinder change?

Yes, because it's an incompressible material, changes in radius will lead to changes in length, $V = \pi a^2 L = \pi r(a)^2 l \Rightarrow l = \lambda L$. Or from the deformation: $l = z(L) = \lambda L$, where $\lambda \neq 1$ shown above.

Problem 2

Consider a system with n particles, n_A of type A and n_B of type B ($n_A + n_B = n$), trying to occupy n sites.

a) Find the probability of any particular configuration.

Solution

$$P_i = \left(\frac{n!}{n_A!n_B!} \right)^{-1} \quad (29)$$

b) Show that the energy of the system for large n is given by

$$F = nk_bT \{ \lambda \log \lambda + (1 - \lambda) \log(1 - \lambda) \} \quad (30)$$

where $\lambda = n_A/n$, k_b and T are the Boltzmann constant and temperature, respectively.

Solution

The total number of micro-configurations $\Omega = \frac{n!}{n_A!n_B!}$. The entropy of the system

$$\begin{aligned} S &= -k_b \sum_{i=1}^{\Omega} P_i \log P_i \equiv k_b \log \Omega = k_b \log \frac{n!}{n_A!n_B!} \\ &= k_b (\log n! - \log n_A! - \log n_B!) \\ &= k_b (n \log n - n_A \log n_A - n_B \log n_B) \\ &= k_b ((n_A + n_B) \log n - n_A \log n_A - n_B \log n_B) \\ &= -k_b (n_A \log n_A/n - n_B \log n_B/n) \\ &= -k_b n \{ \lambda \log \lambda - (1 - \lambda) \log(1 - \lambda) \} \end{aligned} \quad (31)$$

$$\Rightarrow F = -TS = nk_bT \{ \lambda \log \lambda - (1 - \lambda) \log(1 - \lambda) \} \quad (32)$$

Problem 3

Which of the following are Bravais lattices, and what are the lattice vectors?

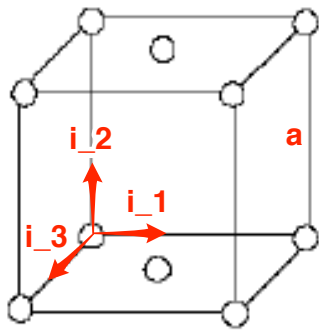
a) Base centered cubic: atoms at the corners of a cube, and at the center of the base and the top.

Solution This is a Bravais lattice with with lattice vectors

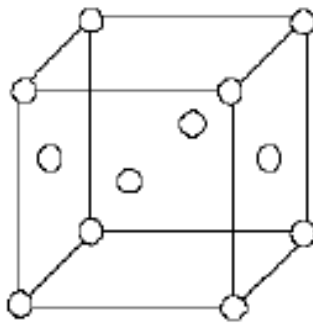
$$\mathbf{e}_1 = \frac{a}{2}(\hat{i}_1 + \hat{i}_3), \quad \mathbf{e}_2 = \frac{a}{2}(\hat{i}_1 - \hat{i}_3), \quad \mathbf{e}_3 = a\hat{i}_2. \quad (33)$$

b) Side centered cubic: atoms at the corners of a cube, and at the center of each vertical face.

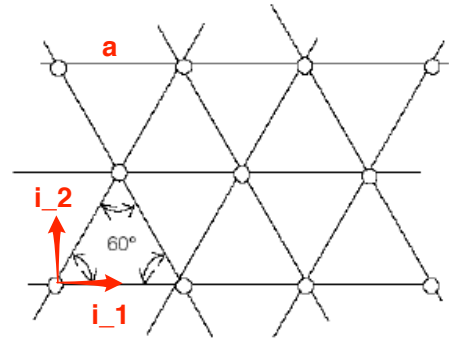
Solution This is not a Bravais lattice, there are no vectors that generate the lattice.



(a) Base centered cubic



(b) Side centered cubic



(c) Triangular lattice (in 2D)

c) Triangular lattice (in 2D).

Solution This is a Bravais lattice with lattice vectors

$$\underline{e}_1 = a\underline{i}_1, \quad \underline{e}_2 = a \left(\frac{1}{2}\underline{i}_1 + \frac{\sqrt{3}}{2}\underline{i}_2 \right). \quad (34)$$