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Spectrum of the Laplace–Beltrami operator on suspensions of toric automorphisms

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Abstract. The spectrum and the eigenbasis of the Laplace–Beltrami operator on the suspensions of toric automorphisms are investigated. A description in terms of solutions of one-dimensional Schrödinger's equation is presented.

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§ 1. Introduction

The well-known problem of the recovery of a Riemannian manifold from the spectrum of the corresponding Laplace–Beltrami operator is commonly believed to be stated in 1966, in [1] as the famous question: "Can one hear the shape of a drum?". This is the problem of equivalence of the concepts of isospectral and isometric manifolds: will manifolds with the same spectrum be isometric? In the general case the answer depends on the geometry of the manifolds [2]. In this connection the description of the spectrum of a Riemannian manifold is of current interest in its own right.

In the present paper we continue the studies of Bolsinov, Taĭmanov, Veselov, and Dullin [3], [4].

In the paper [3] its authors consider geodesic flows on the suspensions of toric automorphisms. One calls a closed manifold M_A^{n+1} the *suspenson of an automorphism* $A\colon \mathbb{T}^n \to \mathbb{T}^n$ if there exists a fibration

$$p\colon M_A^{n+1} \xrightarrow{\overset{A}{\curvearrowright}\atop \mathbb{T}^n} S^1$$

of this manifold over the circle S^1 with fibre $\mathbb{T}^n$ such that the matrix of its monodromy is $A \in \mathrm{SL}(n,\mathbb{Z})$. The manifold M_A^{n+1} has interesting properties: it turns out that there exists on M_A^{n+1} a real-analytic metric with geodesic flow that is Liouville integrable by means of smooth integrals, but has topological entropy distinct from zero.

The integrability problem for the geodesic flow has a quantum analogue: the description of the spectrum and eigenfunctions of the Laplace–Beltrami operator. In [4] one can find a basis in $L_2(M_A^3)$ consisting of eigenfunctions of the Laplace–Beltrami operator and described in terms of solutions of the so-called modified Mathieu's equation.

In the present paper we consider the many-dimensional case $n > 2$, and our main result is the description of the spectrum and the construction of an eigenbasis of

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the Laplace–Beltrami operator in $L_2(M_A^{n+1})$ described in terms of solutions of the one-dimensional Schrödinger's equation.

§ 2. Construction of a Riemannian metric on M_A^{n+1}

One can define the suspension M_A^{n+1} of a toric automorphism as follows. Consider the cylinder $\mathscr{C}^{n+1} = \mathbb{T}^n \times \mathbb{R}$. Let $x = (x_1, \dots, x_n)$ be the standard coordinate variables on $\mathbb{T}^n$ defined modulo 1, and let z be the variable along the line. Consider the action of the group $\mathbb{Z}$ on the cylinder generated by the transformation

$$T_A \colon (x, z) \mapsto (Ax, z + 1), \tag{1}$$

where the integer matrix A defines a toric automorphism $A\colon \mathbb{T}^n \to \mathbb{T}^n$. Then one defines the corresponding manifold M_A^{n+1} as the quotient of the cylinder by this action: $M_A^{n+1} = \mathscr{C}^{n+1}/\mathbb{Z}$.

We shall assume throughout that $A \in \mathrm{SL}(n, \mathbb{Z})$ is a hyperbolic matrix (which means by definition that its eigenvalues λ satisfy the inequality $|\lambda| \neq 1$) with positive spectrum $\mathrm{Spec}(A) \subset \mathbb{R}_+$.

In place of the standard periodic coordinates $(x_1, \dots, x_n)$ on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ we shall use other linear variables on the torus, compatible with the action of the hyperbolic automorphism A. Let $\mathrm{Spec}(A) = \{\lambda_1, \dots, \lambda_k\}$ be the spectrum and let n_α be the multiplicity of the eigenvalue λ_α, $\alpha = 1, \dots, k$. Then in some basis $\{f_i\}_{i=1}^n$ (consisting of eigenvectors and associate vectors) the matrix A has a Jordan normal form

$$A = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{pmatrix},$$

where B_α is a Jordan n_α-block:

$$B_\alpha = \begin{pmatrix} \lambda_\alpha & 1 & \dots & 0 \\ 0 & \lambda_\alpha & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_\alpha \end{pmatrix}.$$

Let $(u_1, \dots, u_n)$ be the variables on the torus corresponding to the basis $\{f_i\}_{i=1}^n$. Note that these are not periodic coordinates: $u=(u_1, \dots, u_n)$ and $\widehat{u}=(\widehat{u}_1, \dots, \widehat{u}_n)$ define the same point in $\mathbb{T}^n$ if and only if $u - \widehat{u} = a_1 e_1 + \dots + a_n e_n$, where $a_i \in \mathbb{Z}$ and $\{e_i\}$ is a basis of the lattice Γ, associated with $\mathbb{T}^n$.

We construct a Riemannian metric on M_A^{n+1} as follows. We start with a metric on the cylinder $\mathscr{C}^{n+1}$. Let

$$ds^2 = g_{ij}(z)\, du^i du^j + dz^2,$$

where

$$G(z) = (g_{ij}(z)) = (e^{-z \ln A})^T G_0 e^{-z \ln A}.$$

Here G_0 is an arbitrary positive-definite symmetric matrix of order n (the matrix of the metric on the zero level $\{z=0\}$), and $\ln A$ is the single-valued real branch of the logarithm, which is well defined because $\operatorname{Spec}(A)\subset\mathbb{R}_+$. Obviously, this metric on the cylinder is invariant under the above-described action of the discrete group, therefore it reduces to a metric on the quotient space $M_A^{n+1}=\mathscr{C}^{n+1}/\mathbb{Z}$.

§ 3. Laplace–Beltrami operator on M_A^{n+1}

The Laplace–Beltrami operator is the operator $\Delta=\operatorname{div}\operatorname{grad}$, which has the following representation on a Riemannian manifold:

$$\Delta=\frac{1}{\sqrt{\det G}}\,\partial_i(g^{ij}\sqrt{\det G}\,\partial_j);$$

here ∂_i is the partial derivative with respect to the ith coordinate variable, G is the metric tensor, and the g^{ij} are the components of the inverse metric tensor in the local coordinate chart.

In our case $\det G=\det G_0$. Since A is a unimodular matrix, the Laplace–Beltrami operator on M_A^{n+1} has the following representation in the variables $(u_1,\dots,u_n,z)$:

$$\Delta=g^{ij}(z)\frac{\partial^2}{\partial u_i\partial u_j}+\frac{\partial^2}{\partial z^2}\,,$$

where

$$G^{-1}(z)=(g^{ij}(z))=e^{z\ln A}G_0^{-1}(e^{z\ln A})^T.$$

Calculations show that

$$G^{-1}(z)=\begin{pmatrix} e^{z\ln B_1}G_0^{11}e^{z\ln B_1^T} & \dots & e^{z\ln B_1}G_0^{1k}e^{z\ln B_k^T}\\ \vdots & \ddots & \vdots\\ e^{z\ln B_k}G_0^{k1}e^{z\ln B_1^T} & \dots & e^{z\ln B_k}G_0^{kk}e^{z\ln B_k^T}\end{pmatrix},$$

where $G_0^{-1}=(G_0^{\alpha\beta})$ is the decomposition into submatrices induced by the decomposition of A into Jordan blocks.

The matrix $e^{z\ln B_\alpha}G_0^{\alpha\beta}e^{z\ln B_\beta^T}\in\operatorname{Mat}(n_\alpha,n_\beta)$ has the following form:

$$e^{z\ln B_\alpha}G_0^{\alpha\beta}e^{z\ln B_\beta^T}=e^{z\ln\lambda_\alpha\lambda_\beta}P_{\alpha\beta}(z),\qquad P_{\alpha\beta}(z)\in\operatorname{Mat}(n_\alpha,n_\beta).$$

The entries $p_{ij}^{\alpha\beta}$ of the matrix $P_{\alpha\beta}(z)$ are polynomials of z; we arrange their degrees $\deg p_{ij}^{\alpha\beta}(z)=n_\alpha+n_\beta-i-j$ in the following table:

$n_\alpha+n_\beta-2$	$\dots$	$\dots$	$\dots$	$n_\alpha-1$
$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$\vdots$	$\ddots$	4	3	2
$\vdots$	$\ddots$	3	2	1
$n_\beta-1$	$\dots$	2	1	0

We see that the Laplace–Beltrami operator on M_A^{n+1} has the following form in the variables $(u_1,\dots,u_n,z)$:

$$\Delta=\sum_{\alpha,\beta=1}^{(k,k)}\sum_{i,j=1}^{(n_\alpha,n_\beta)} e^{z\ln\lambda_\alpha\lambda_\beta}p_{ij}^{\alpha\beta}(z)\frac{\partial^2}{\partial u_\xi\partial u_\eta}+\frac{\partial^2}{\partial z^2}\,, \tag{2}$$

$$\xi=\xi(\alpha,i)=\sum_{s=1}^{\alpha-1}n_s+i,\qquad \eta=\eta(\beta,j)=\sum_{s=1}^{\beta-1}n_s+j.$$

Here the subscripts α and β define a submatrix of $G^{-1}(z)$ and the indices i and j define an entry of this submatrix.

§ 4. Spectrum and eigenfunctions of the Laplace–Beltrami operator

The integrability problem for the geodesic flow of a Riemannian metric on a manifold has a quantum analogue, the description of the spectrum and the eigenfunctions of the Laplace–Beltrami operator corresponding to this metric

$$-\Delta\Psi=\mathscr{E}\Psi. \tag{3}$$

In our case $\Psi\in L_2(M_A^{n+1})$ and Δ is the above-described Laplace–Beltrami operator on $L_2(M_A^{n+1})$.

Since the coefficients Δ depend only on z, the natural approach is to separate variables and to seek eigenfunctions of Δ in the following form:

$$\Psi_\gamma(u,z)=e^{2\pi i\langle\gamma,u\rangle}F(z), \tag{4}$$

where $\gamma=\sum_{i=1}^n\gamma_ie_i^*\in\Gamma^*$, $\gamma_i\in\mathbb{Z}$, is an element of the dual lattice of the torus, and $F\in L_2(\mathbb{R})$. We observe that pairing $\langle\gamma,u\rangle$ is defined modulo $\mathbb{Z}$:

$$\langle\gamma,u\rangle=\left\langle\gamma,u+\sum_{i=1}^n a_ie_i\right\rangle=\langle\gamma,u\rangle+\sum_{i=1}^n\gamma_ia_i.$$

However, $e^{2\pi i\mathbb{Z}}=1$ and therefore the function $\Psi_\gamma(u,z)$ is well defined on M_A^{n+1}.

Substituting now (4) in equations (3), (2) we obtain

$$-\frac{d^2F(z)}{dz^2}+Q_\gamma(z)F(z)=\mathscr{E}F(z), \tag{5}$$

$$Q_\gamma(z)=(2\pi)^2\sum_{\alpha,\beta=1}^{(k,k)}\sum_{i,j=1}^{(n_\alpha,n_\beta)} e^{z\ln\lambda_\alpha\lambda_\beta}p_{ij}^{\alpha\beta}(z)\langle\gamma,f_\xi\rangle\langle\gamma,f_\eta\rangle. \tag{6}$$

The problem of the spectrum and the eigenfunctions of the Laplace–Beltrami operator on the Sol-manifold M_A^{n+1} reduces in this way to the one-dimensional Schrödinger's equation (5) on the line with potential $Q_\gamma(z)$. For $n=2$ equation (5) reduces to the so-called modified Mathieu's equation [4].

The properties of the one-dimensional Schrödinger's equation (5) are well studied. As is known [5], if $Q_\gamma(z)\to+\infty$ as $|z|\to\infty$, then there exists a complete orthonormal system of eigenfunctions $F_k(z)$, $k\in\mathbb{N}$, of the operator $-d^2/dz^2+Q_\gamma(z)$ belonging to $L^2(\mathbb{R})$ and with eigenvalues $\mathscr{E}_k$ approaching $+\infty$ as $k\to\infty$.

Lemma 1. *If $\gamma \neq 0$, then $Q_\gamma(z) \to +\infty$ as $|z| \to \infty$.*

Proof. We observe first of all that

$$Q_\gamma(z) = (2\pi)^2 \cdot (\langle\gamma, f_1\rangle, \dots, \langle\gamma, f_n\rangle) G^{-1}(z) \begin{pmatrix} \langle\gamma, f_1\rangle \\ \vdots \\ \langle\gamma, f_n\rangle \end{pmatrix}.$$

Hence $Q_\gamma(z) > 0$ for $\gamma \neq 0$ because the metric is positive-definite and one has $(\langle\gamma, f_1\rangle, \dots, \langle\gamma, f_n\rangle) \neq 0$ for $\gamma \neq 0$.

Let $z \to +\infty$ (the case when $z \to -\infty$ is similar). It follows from (6) that the potential $Q_\gamma(z)$ is a linear combination of functions of the form $e^{\mu z} q(z)$ with $\mu \in \mathbb{R}$ and $q(z)$ a polynomial . Hence it is sufficient for the proof of the lemma to demonstrate that this linear combination (after collecting similar terms) contains a term with $\mu > 0$. Indeed, in that case such a term approaches $\pm\infty$ depending on the polynomial factor, so that $Q_\gamma(z) \to \pm\infty$. However, $Q_\gamma(z) > 0$, and therefore only the convergence $Q_\gamma(z) \to +\infty$ is possible. We claim that there exists such a term.

The space $\mathbb{R}^n$ of the hyperbolic operator A is the sum of two invariant subspaces:

$$\mathbb{R}^n = V_A^- \oplus V_A^+, \qquad V_A^- = \bigoplus_{\lambda<1} V_\lambda, \quad V_A^+ = \bigoplus_{\lambda>1} V_\lambda.$$

Let V_λ be the root subspace corresponding to λ.

Lemma 2. *If $A \in \mathrm{SL}(n, \mathbb{Z})$ is a hyperbolic matrix, then*

$$\mathrm{Ann}(V_A^+) \cap \Gamma^* = 0.$$

Proof. Since $\mathbb{R}^n = V_A^- \oplus V_A^+$, it follows that $\mathrm{Ann}(V_A^+) = (V_A^-)^*$. Hence

$$\mathrm{Ann}(V_A^+) \cap \Gamma^* = (V_A^-)^* \cap \Gamma^* = (V_A^- \cap \Gamma)^*.$$

Since $V_A^- \cap \Gamma$ is a subgroup of the additive group $\mathbb{R}^n$ and is discrete as a subset, $V_A^- \cap \Gamma$ is a lattice [6]. Moreover, $V_A^- \cap \Gamma$ is A-invariant. Thus, $V_A^- \cap \Gamma$ is an A-invariant sublattice of Γ.

Assume that there exists a non-trivial $v \in V_A^- \cap \Gamma$. Then $A^n v \in V_A^- \cap \Gamma$ for arbitrary $n \in \mathbb{N}$. On the one hand, $V_A^- \cap \Gamma$ is a discrete set and therefore there exists a punctured neighbourhood U of the origin in $\mathbb{R}^n$ containing no points from $V_A^- \cap \Gamma$. On the other hand, V_A^- consists of vectors v such that $A^n v \to 0$ as $n \to \infty$, and therefore $A^n v \in U$ for each n starting with some exponent, which is a contradiction. Hence $V_A^- \cap \Gamma = 0$, and therefore $\mathrm{Ann}(V_A^+) \cap \Gamma^* = 0$. The proof of Lemma 2 is complete.

By Lemma 2, for each point $\gamma \neq 0$ in the dual lattice the basis $\{f_i\}_{i=1}^n$ contains a vector $f_0 \in V_{\lambda_0} \subset V_A^+$ such that $\langle\gamma, f_0\rangle \neq 0$. We shall assume without loss of generality that $f_0 = f_1$ and $\lambda_0 = \lambda_1$. Then

$$Q_\gamma(z) = (2\pi)^2 e^{2z \ln \lambda_1} p_{11}^{11}(z) \langle\gamma, f_1\rangle^2 + \cdots.$$

It remains to observe that the polynomial $p_{11}^{11}(z)$ is distinct from the identically zero function because $p_{11}^{11}(0) = (G_0^{-1})_{11} > 0$ as a diagonal element of the positive-definite matrix G_0^{-1}. The proof of Lemma 1 is complete.

Thus, we have associated with each element $\gamma \in \Gamma^* \setminus \{0\}$ a series of eigenfunctions and eigenvalues of the Laplace–Beltrami operator on the cylinder $\mathscr{C}^{n+1}$:

$$\gamma \mapsto (\Psi_{\gamma,k}, \mathscr{E}_{\gamma,k}),$$
$$\Psi_{\gamma,k}(u,z) = e^{2\pi i \langle \gamma, u\rangle} F_{\gamma,k}(z).$$

However, the $\Psi_{\gamma,k}$ are not well-defined functions on $M_A^{n+1} = \mathscr{C}^{n+1}/\mathbb{Z}$ because they are not invariant under the action (1) of the discrete group on the cylinder. It is natural to average these functions over the action, that is, to consider in place of $\Psi_{\gamma,k}$ the series

$$\Psi_{\gamma,k}(u,z) \mapsto \sum_{n\in\mathbb{Z}} \Psi_{\gamma,k}(A^n u, z+n) =: \widetilde{\Psi}_{\gamma,k}(u,z). \tag{7}$$

The transformation $(u,z) \mapsto (Au, z+1)$ is an isometry and therefore preserves the Laplace operator. Hence the function $\Psi^{(n)}_{\gamma,k}(u,z) := \Psi_{\gamma,k}(A^n u, z+n)$ solves equation (3) with the same eigenvalue $\mathscr{E}_{\gamma,k}$.

We now write down the chain of equalities

$$\begin{aligned}\Delta\widetilde{\Psi}_{\gamma,k}(u,z) &= \Delta\sum_{n\in\mathbb{Z}} \Psi_{\gamma,k}(A^n u, z+n) = \sum_{n\in\mathbb{Z}} \Delta\Psi_{\gamma,k}(A^n u, z+n) \\ &= \sum_{n\in\mathbb{Z}} \mathscr{E}_{\gamma,k}\Psi_{\gamma,k}(A^n u, z+n) = \mathscr{E}_{\gamma,k}\widetilde{\Psi}_{\gamma,k}(u,z).\end{aligned} \tag{8}$$

To make these formal equalities meaningful one must show that the series (7) converges to a well-defined function on M_A^{n+1} and that the Laplace–Beltrami operator commutes with summation.

As is known [5], each solution of the equation

$$-\frac{d^2 f(z)}{dz^2} + v(z)f(z) = 0, \qquad v(z) \to +\infty \quad \text{as } |z| \to +\infty,$$

has one of the following properties:

(1) for each $\alpha > 0$ there exists M such that $|f(z)| \geqslant e^{\alpha|z|}$ for $|z| \geqslant M$;

(2) for each $\alpha > 0$ there exists M such that $|f(z)| \leqslant e^{-\alpha|z|}$ for $|z| \geqslant M$.

In our case (5), by Lemma 1 we obtain $v(z) = Q_\gamma(z) - \mathscr{E} \to +\infty$ as $|z| \to +\infty$. In addition, all the eigenfunctions $F_{\gamma,k}(z)$ belong to $L^2(\mathbb{R})$, therefore for the one-dimensional Schrödinger's equation with increasing potential only case (2) can occur. Thus, there exists $M = M(\gamma,k)$ such that for all $|z| > M$,

$$|F_{\gamma,k}(z)| \leqslant e^{-|z|}. \tag{9}$$

We now establish several simple results substantiating (8).

Lemma 3. *The series*

$$\sum_{n\in\mathbb{Z}} \Psi^{(n)}_{\gamma,k}(u,z) = \sum_{n\in\mathbb{Z}} \Psi_{\gamma,k}(A^n u, z+n)$$

converges pointwise in the cylinder $\mathscr{C}^{n+1} = \mathbb{T}^n \times \mathbb{R}$.

Proof. Let $(u_0, z_0) \in \mathscr{C}^{n+1}$. Then

$$\left|\sum_{n\in\mathbb{Z}} \Psi_{\gamma,k}(A^n u_0, z_0+n)\right| = \left|\sum_{n\in\mathbb{Z}} e^{2\pi i\langle\gamma, A^n u_0\rangle} F_{\gamma,k}(z_0+n)\right| \leqslant \sum_{n\in\mathbb{Z}} |F_{\gamma,k}(z_0+n)|.$$

Using the estimate (9) we obtain

$$\sum_{n\in\mathbb{Z}} |F_{\gamma,k}(z_0+n)| \leqslant (e^{z_0}+e^{-z_0})e^{-M-1}\frac{e}{e-1} + \sum_{n=-M}^{M} |F_{\gamma,k}(z_0+n)|$$

for sufficiently large M. This inequality proves Lemma 3.

Thus, the function $\widetilde{\Psi}_{\gamma,k}$ is well defined on $\mathscr{C}^{n+1}$. It is moreover easy to see that $\widetilde{\Psi}_{\gamma,k}$ is also a function on the quotient $\mathscr{C}^{n+1}/\mathbb{Z}$ because it is invariant under the transformation $(u,z) \mapsto (Au, z+1)$.

Lemma 4. (1) *The series* $\sum_{n\in\mathbb{Z}} \Psi^{(n)}_{\gamma,k}(u,z)$ *converges uniformly on* $\mathbb{T}^n \times [0,1]$;

(2) $\Psi^{(n)}_{\gamma,k}(u,z) \in L_2(\mathscr{C}^{n+1}) \subset L_2(\mathbb{T}^n \times [0,1])$;

(3) $\widetilde{\Psi}_{\gamma,k} \in L_2(M_A^{n+1})$.

Proof. For the proof of part (1) it is necessary and sufficient to show that the remainder term of the series converges to zero uniformly on $\mathbb{T}^n \times [0,1]$. As follows from the proof of Lemma 3, the remainder has the estimate

$$\sum_{|n|>M} \Psi^{(n)}_{\gamma,k}(u,z) \leqslant (e^z+e^{-z})\frac{e^{-M}}{e-1}\,.$$

This inequality holds for all $z \in [0,1]$ if M is sufficiently large. The right-hand side approaches zero as $M \to \infty$, which proves the first assertion of the lemma.

Now,

$$\int_{\mathbb{T}^n\times[0,1]} |\Psi^{(n)}_{\gamma,k}|^2\,du\,dz \leqslant \int_{\mathscr{C}^{n+1}} |\Psi^{(n)}_{\gamma,k}|^2\,du\,dz = \operatorname{area}(\mathbb{T}^n)\int_{\mathbb{R}} |F_{\gamma,k}(z+n)|^2\,dz$$
$$= \operatorname{area}(\mathbb{T}^n)\cdot\|F_{\gamma,k}\|^2_{L_2(\mathbb{R})} < \infty.$$

By the first two assertions of the lemma

$$\widetilde{\Psi}_{\gamma,k} \in L_2(\mathbb{T}^n \times [0,1]).$$

The space $M_A^{n+1} = \mathscr{C}^{n+1}/\mathbb{Z}$ can be obtained from $\mathbb{T}^n \times [0,1]$ by identifying the boundary tori by means of the automorphism with matrix $A \in \mathrm{SL}(n,\mathbb{Z})$. Hence

$$\int_{M_A^{n+1}} |\widetilde{\Psi}_{\gamma,k}|^2 \leqslant \int_{\mathbb{T}^n\times[0,1]} |\widetilde{\Psi}_{\gamma,k}|^2,$$

which completes the proof of Lemma 4.

Lemmas 3, 4, and the equality $\Delta\Psi^{(n)}_{\gamma,k}(u,z) = \mathscr{E}_{\gamma,k}\Psi^{(n)}_{\gamma,k}(u,z)$ show that the Laplace–Beltrami operator commutes with summation.

We have thus substantiated the chain of equalities (8) and the transition from the functions $\Psi_{\gamma,k}$ on the cylinder to the functions $\widetilde{\Psi}_{\gamma,k}$ on the quotient space $M^{n+1}_A = \mathscr{C}^{n+1}/\mathbb{Z}$.

In this way we associate with each element $\gamma \in \Gamma^* \setminus \{0\}$ of the dual lattice of the torus $\mathbb{T}^n$ a series of eigenvalues and 'correct' eigenfunctions of the Laplace–Beltrami operator on $L_2(M^{n+1}_A)$:

$$\gamma \mapsto (\widetilde{\Psi}_{\gamma,k}(u,z), \mathscr{E}_{\gamma,k}),$$
$$\widetilde{\Psi}_{\gamma,k}(u,z) = \sum_{n\in\mathbb{Z}} \Psi_{\gamma,k}(A^n u, z+n) = \sum_{n\in\mathbb{Z}} e^{2\pi i\langle\gamma, A^n u\rangle} F_{\gamma,k}(z+n).$$

Consider now the natural action of the cyclic subgroup $\{A^*\} \subset \mathrm{SL}(n,\mathbb{Z})$ on Γ^*. Let $[\gamma]$ be the orbit of this action:

$$[\gamma] = \{(A^*)^n\gamma : n \in \mathbb{Z}\}.$$

Lemma 5. *If $\gamma_1, \gamma_2 \in \Gamma^* \setminus \{0\}$ are points in the same orbit of the action $\{A^*\} : \Gamma^*$, then the corresponding eigenvalues and eigenfunctions are the same.*

Proof. Since γ_1, γ_2 belong to the same orbit, there exists $N \in \mathbb{Z}$ such that $\gamma_2 = (A^*)^N\gamma_1$. Then

$$\widetilde{\Psi}_{\gamma_2,k}(u,z) = \sum_{n\in\mathbb{Z}} \Psi_{(A^*)^N\gamma_1,k}(A^n u, z+n).$$

Lemma 6. $\Psi_{A^*\gamma,k}(u,z) = \Psi_{\gamma,k}(Au, z+1)$.

Proof. We have

$$\Psi_{A^*\gamma,k}(u,z) = e^{2\pi i\langle A^*\gamma, u\rangle} F_{A^*\gamma,k}(z) = e^{2\pi i\langle\gamma, Au\rangle} F_{A^*\gamma,k}(z).$$

The function $F_{A^*\gamma,k}$ is the kth solution of the one-dimensional Schrödinger's equation with potential

$$\begin{aligned}
Q_{A^*\gamma}(z) &= (2\pi)^2 g^{ij}(z)\langle A^*\gamma, f_i\rangle\langle A^*\gamma, f_j\rangle \\
&= (2\pi)^2\big(\langle A^*\gamma, f_1\rangle, \dots, \langle A^*\gamma, f_n\rangle\big)\cdot G^{-1}(z)\cdot\big(\langle A^*\gamma, f_1\rangle, \dots, \langle A^*\gamma, f_n\rangle\big)^T \\
&= (2\pi)^2\big(\langle \gamma, Af_1\rangle, \dots, \langle \gamma, Af_n\rangle\big)\cdot G^{-1}(z)\cdot\big(\langle \gamma, Af_1\rangle, \dots, \langle \gamma, Af_n\rangle\big)^T \\
&= (2\pi)^2\big(\langle \gamma, f_1\rangle, \dots, \langle \gamma, f_n\rangle\big)\cdot AG^{-1}(z)A^T\cdot\big(\langle \gamma, f_1\rangle, \dots, \langle \gamma, f_n\rangle\big)^T \\
&= (2\pi)^2\big(\langle \gamma, f_1\rangle, \dots, \langle \gamma, f_n\rangle\big)\cdot G^{-1}(z+1)\cdot\big(\langle \gamma, f_1\rangle, \dots, \langle \gamma, f_n\rangle\big)^T \\
&= (2\pi)^2 g^{ij}(z+1)\langle\gamma, f_i\rangle\langle\gamma, f_j\rangle = Q_\gamma(z+1).
\end{aligned}$$

That is, $F_{A^*\gamma,k}(z)$ is the kth solution of the one-dimensional Schrödinger's equation with potential $Q_{A^*\gamma}(z) = Q_\gamma(z+1)$. Hence

$$F_{A^*\gamma,k}(z) = F_{\gamma,k}(z+1).$$

We finally obtain

$$\Psi_{A^*\gamma,k}(u,z) = e^{2\pi i\langle \gamma, Au\rangle} F_{\gamma,k}(z+1) = \Psi_{\gamma,k}(Au, z+1).$$

The proof of Lemma 6 is complete.

By Lemma 6,

$$\begin{aligned}\widetilde{\Psi}_{\gamma_2,k}(u,z) &= \sum_{n\in\mathbb{Z}} \Psi_{(A^*)^N\gamma_1,k}(A^n u, z+n)\\ &= \sum_{n\in\mathbb{Z}} \Psi_{\gamma_1,k}(A^{n+N}u, z+n+N) = \widetilde{\Psi}_{\gamma_1,k}(u,z).\end{aligned}$$

The proof of Lemma 5 is also now complete.

We see that the eigenfunctions and the eigenvalues of the Laplace–Beltrami operator are parametrized by entire orbits rather than points in the dual lattice:

$$[\gamma] \mapsto (\Psi_{[\gamma],k}, \mathscr{E}_{[\gamma],k}),$$
$$\Psi_{[\gamma],k} := \widetilde{\Psi}_{\gamma,k}.$$

We consider separately the case $\gamma = 0$. The one-dimensional Schrödinger's equation (5) now takes the very simple form

$$\frac{d^2F(z)}{dz^2} + \mathscr{E}F(z) = 0,$$

so that for $\gamma = 0$ we obtain the following system of eigenfunctions and eigenvalues:

$$(1,0), \quad (\cos k\pi z, (\pi k)^2), \quad (\sin k\pi z, (\pi k)^2).$$

Theorem 1. *The system of functions* $\{\Psi_{[\gamma],k} : \gamma\in\Gamma^*\setminus\{0\}\}\cup\{1,\ \cos k\pi z,\ \sin k\pi z\}$ *with* $k\in\mathbb{N}$ *forms an eigenbasis of the Laplace–Beltrami operator in* $L_2(M_A^{n+1})$. *The eigenvalue* $\mathscr{E}_{[\gamma],k}$ *corresponding to a function* $\Psi_{[\gamma],k}$ *is an eigenvalue of the Schrödinger operator on the line with potential* $Q_\gamma(z)$ (6).

Proof. It is obvious that these functions are orthogonal and independent. It remains to verify that the system of functions is complete in $L_2(M_A^{n+1})$.

As is known [7], [8], if M is a compact Riemannian manifold, then there exists in $L_2(M)$ an orthonormal basis of infinitely smooth eigenfunctions of the Laplace–Beltrami operator (because this operator is elliptic). Hence it is sufficient for the completeness to show that if $\Phi\in L^2(M_A^{n+1})$ is a smooth function orthogonal to all the functions in our system, then it is identically equal to zero [9].

Similarly to the case of $S^1 = \mathbb{T}^1$, the system of functions

$$\{\varphi_\gamma(u) = e^{2\pi i\langle\gamma,u\rangle} : \gamma\in\Gamma^*\}$$

makes up a complete orthogonal family in $L_2(\mathbb{T}^n)$, $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ [9]. Hence each $\Phi\in L_2(M_A^{n+1})$ can be represented as a series

$$\Phi(u,z) = \sum_{\gamma\in\Gamma^*} e^{2\pi i\langle\gamma,u\rangle} c_\gamma(z).$$

This is the Fourier series of Φ with respect to the orthogonal system $\{\varphi_\gamma\}$. The Fourier coefficients c_γ are defined by the formula

$$c_\gamma(z) = \frac{1}{\|\varphi_\gamma\|^2_{L^2(\mathbb{T}^n)}} \int_{\mathbb{T}^n} \Phi(u,z)\overline{\varphi_\gamma(u)}\, d\mu.$$

Without loss of generality we can regard Φ as a function on the cylinder $\mathscr{C}^{n+1}$. Then c_γ will be a smooth function on the line.

Lemma 7. *If* $\gamma \neq 0$, *then* $c_\gamma \in L_2(\mathbb{R})$.

Proof. Since $\Phi \in L^2(M_A^{n+1})$, it follows that $\Phi(Au, z+1) = \Phi(u,z)$. Hence

$$\begin{aligned}\sum_{\gamma\in\Gamma^*} e^{2\pi i\langle\gamma,u\rangle} c_\gamma(z) &= \sum_{\gamma\in\Gamma^*} e^{2\pi i\langle\gamma,Au\rangle} c_\gamma(z+1) = \sum_{\gamma\in\Gamma^*} e^{2\pi i\langle A^*\gamma,u\rangle} c_\gamma(z+1) \\ &= \sum_{\gamma\in\Gamma^*} e^{2\pi i\langle\gamma,u\rangle} c_{(A^*)^{-1}\gamma}(z+1).\end{aligned}$$

We see that the Fourier coefficients satisfy the condition

$$c_\gamma(z+1) = c_{A^*\gamma}(z),$$

or, more generally,

$$c_\gamma(z+n) = c_{(A^*)^n\gamma}(z), \qquad n \in \mathbb{Z}.$$

If $\gamma \neq 0$, then it follows by the hyperbolicity of A that $(A^*)^n\gamma \to \infty$ as $n \to \pm\infty$. It is well known that the more rapidly the Fourier coefficients approach zero the better are the differential properties of the function in question. In particular, if a function has k derivatives, then its Fourier coefficients c_n satisfy the estimate $c_n = o(1/|n|^k)$ as $n \to \pm\infty$ [10]. Hence in our case we obtain

$$c_{(A^*)^n\gamma}(z) = o\biggl(\frac{1}{|(A^*)^n\gamma|^k}\biggr) = o\biggl(\frac{1}{(\Lambda^n)^k}\biggr) \quad \text{as } n \to \pm\infty,$$

where Λ is the largest eigenvalue of A. By the hyperbolicity $\Lambda > 1$, therefore it follows by the above equality that the coefficients c_γ approach zero at infinity sufficiently rapidly to belong to $L_2(\mathbb{R})$. The proof of Lemma 7 is complete.

Now let Φ be a function orthogonal to all the $\Psi_{[\gamma_0],k}$, $\gamma_0 \in \Gamma^* \setminus \{0\}$. Then

$$
\begin{aligned}
0 &= \langle \Phi, \Psi_{[\gamma_0],k} \rangle_{L^2(M_A^{n+1})} = \int_{M_A^{n+1}} \Phi(u,z) \overline{\Psi}_{[\gamma_0],k}(u,z)\, d\sigma \\
&= \int_{M_A^{n+1}} \sum_{\gamma\in\Gamma^*} e^{2\pi i \langle \gamma, u\rangle} c_\gamma(z) \sum_{n\in\mathbb{Z}} e^{-2\pi i \langle (A^*)^n \gamma_0, u\rangle} \overline{F}_{(A^*)^n\gamma_0,k}(z)\, d\sigma \\
&= \int_0^1 \sum_{\gamma\in\Gamma^*, n\in\mathbb{Z}} \biggl(\int_{\mathbb{T}^n} e^{2\pi i \langle \gamma, u\rangle} e^{-2\pi i \langle (A^*)^n \gamma_0, u\rangle}\, d\mu \biggr) c_\gamma(z) \overline{F}_{(A^*)^n\gamma_0,k}(z)\, dz \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} \biggl(\int_{\mathbb{T}^n} e^{2\pi i \langle (A^*)^n \gamma_0, u\rangle} e^{-2\pi i \langle (A^*)^n \gamma_0, u\rangle}\, d\mu \biggr) c_{(A^*)^n\gamma_0}(z) \overline{F}_{(A^*)^n\gamma_0,k}(z)\, dz \\
&= \operatorname{area}(\mathbb{T}^n) \int_0^1 \sum_{n\in\mathbb{Z}} c_{(A^*)^n\gamma_0}(z) \overline{F}_{(A^*)^n\gamma_0,k}(z)\, dz \\
&= \operatorname{area}(\mathbb{T}^n) \int_0^1 \sum_{n\in\mathbb{Z}} c_{\gamma_0}(z+n) \overline{F}_{\gamma_0,k}(z+n)\, dz \\
&= \operatorname{area}(\mathbb{T}^n) \int_{\mathbb{R}} c_{\gamma_0}(z) \overline{F}_{\gamma_0,k}(z)\, dz = \operatorname{area}(\mathbb{T}^n) \cdot \langle c_{\gamma_0}, F_{\gamma_0,k} \rangle_{L_2(R)}.
\end{aligned}
$$

Thus, the Fourier coefficient c_{γ_0} belongs to $L_2(\mathbb{R})$ for $\gamma_0 \neq 0$ and is orthogonal to all the functions $F_{\gamma_0,k}$, which form an orthonormal basis in $L_2(\mathbb{R})$. Hence $c_{\gamma_0} \equiv 0$ for $\gamma_0 \neq 0$, therefore the function Φ has the form $\Phi(u,z) = c_0(z)$, where c_0 is a 1-periodic function. Using now the orthogonality of Φ to all the functions $\cos k\pi z$, $\sin k\pi z$, $k \in \mathbb{N}$, which make up a complete system in $L_2(S^1)$ we obtain that $c_0 \equiv 0$. This proves the completeness and completes the proof of Theorem 1.

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