

Navier-Stokes Equations

In cylindrical coordinates, (r, θ, z) , the continuity equation for an incompressible fluid is

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (u_\theta) + \frac{\partial u_z}{\partial z} = 0$$

In cylindrical coordinates, (r, θ, z) , the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity, μ , and density, ρ , are

$$\rho \left[\frac{D u_r}{D t} - \frac{u_\theta^2}{r} \right] = -\frac{\partial p}{\partial r} + f_r + \mu \left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right]$$

$$\rho \left[\frac{D u_\theta}{D t} + \frac{u_\theta u_r}{r} \right] = -\frac{1}{r} \frac{\partial p}{\partial \theta} + f_\theta + \mu \left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right]$$

$$\rho \frac{D u_z}{D t} = -\frac{\partial p}{\partial z} + f_z + \mu \nabla^2 u_z$$

where u_r, u_θ, u_z are the velocities in the r, θ, z cylindrical coordinate directions, p is the pressure, f_r, f_θ, f_z are the body force components in the r, θ, z directions and the operators D/Dt and ∇^2 are

$$\frac{D}{D t} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and for an incompressible, Newtonian fluid

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} \quad ; \quad \sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial u_z}{\partial z}$$

$$\sigma_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad ; \quad \sigma_{rz} = \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)$$

$$\sigma_{\theta z} = \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right)$$

In spherical coordinates, (r, θ, ϕ) , the continuity equation for an incompressible fluid is :

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} = 0$$

In spherical coordinates, (r, θ, ϕ) , the Navier-Stokes equations of motion for an incompressible fluid with uniform viscosity are:

$$\rho \left[\frac{Du_r}{Dt} - \frac{u_\theta^2 + u_\phi^2}{r} \right] = -\frac{\partial p}{\partial r} + f_r + \mu \left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{2u_\theta \cot \theta}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right]$$

$$\rho \left[\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r} - \frac{u_\phi^2 \cot \theta}{r} \right] = -\frac{1}{r} \frac{\partial p}{\partial \theta} + f_\theta + \mu \left[\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin \theta \sin \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right]$$

$$\rho \left[\frac{Du_\phi}{Dt} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi \cot \theta}{r} \right] = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + f_\phi + \mu \left[\nabla^2 u_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2 \sin \theta \sin \theta} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right]$$

where u_r, u_θ, u_ϕ are the velocities in the r, θ, ϕ directions, p is the pressure, ρ is the fluid density and f_r, f_θ, f_ϕ are the body force components. The Lagrangian or material derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial}{\partial \phi}$$

and the Laplacian operator is

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta \sin \theta} \frac{\partial^2}{\partial \phi^2}$$

and for an incompressible, Newtonian fluid

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} \quad ; \quad \sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)$$

$$\sigma_{\phi\phi} = -p + 2\mu \left(\frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \right)$$

$$\sigma_{r\theta} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \quad ; \quad \sigma_{r\phi} = \mu \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{u_\phi}{r} \right) \right)$$

$$\sigma_{\theta\phi} = \mu \left(\frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{u_\phi}{\sin \theta} \right) \right)$$