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The maximum number of 3- and 4-cliques within a planar maximally filtered graph

Jenna Birch [a], Athanasios A. Pantelous [a,b,*], Konstantin Zuev [b,1]

[a] Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, United Kingdom

[b] Institute for Risk and Uncertainty, University of Liverpool, United Kingdom

HIGHLIGHTS

- Embeddings of n-vertex triangulations in the form of maximal planar graphs.
- Applying the generating operations proposed by Eberhard to construct the maximal planar graphs.
- Any maximal planar graph can be transformed to a standard spherical triangulation.
- The standard spherical triangulation structure always contains the maximum number of 3- and 4-cliques.

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ABSTRACT

Planar Maximally Filtered Graphs (PMFG) are an important tool for filtering the most relevant information from correlation based networks such as stock market networks. One of the main characteristics of a PMFG is the number of its 3- and 4-cliques. Recently in a few high impact papers it was stated that, based on heuristic evidence, the maximum number of 3- and 4-cliques that can exist in a PMFG with n vertices is $3n - 8$ and $n - 4$ respectively. In this paper, we prove that this is indeed the case.

1. Introduction

In recent years there has been increasing interest in how we can model complex systems using network theory. These can include information, technological and biological systems [1], social networks [2] and financial markets [3,4]. In particular, a network based approach of studying complex systems has become very popular in econophysics [5], an interdisciplinary research field that studies economic and financial phenomena. One of the important and fundamental problems in this approach is to filter the most relevant information from financial networks. As a result traditional algorithms from network theory have been adapted and some new methods have been introduced. In 1999 Mantegna [6] introduced a method for finding a hierarchical arrangement for a portfolio of stocks by extracting the Minimum Spanning Tree (MST) from the complete network of correlations of daily closing price returns for US stocks. This work has developed to include Asset Graphs (AG) as introduced in Refs. [7,8] and Threshold Networks [9,10]. Building from the work by Mantegna [6] with the MST, one of the most recent developments was an algorithm proposed by Tumminello et al. [11] where the complete network can be

* Correspondence to: Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences and Institute for Risk and Uncertainty, University of Liverpool, Peach Street, L697ZL Liverpool, United Kingdom. Tel.: +44 0 151 794 5079.

E-mail addresses: j.l.birch@student.liv.ac.uk (J. Birch), a.pantelous@liverpool.ac.uk, aap@liv.ac.uk (A.A. Pantelous), zuev@liv.ac.uk (K. Zuev).

1 Tel.: +44 0 151 794 5263.

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filtered at a chosen level, by varying the genus of the resulting filtered graph. So if a graph is embedded on a surface with genus $= g$, as g increases the resulting graph becomes more complex and so reveals more information about the clusters formed, while keeping the same hierarchical tree as the corresponding MST. The simplest form of this graph is the Planar Maximally Filtered Graph (PMFG), on surface $g = 0$.

The PMFG is constructed in a similar way to the MST. Specifically, for a graph with n vertices, a weighted edge is associated with all paired vertices where the value of the edge is the similarity coefficient between the two vertices. The weighted edges $u_1, u_2, \ldots, u_e$ are placed in descending order $u_{(1)}, u_{(2)}, \ldots, u_{(e)}$. The first edge $u_{(1)}$ is selected and a graph is constructed with $u_{(1)}$ and the two vertices that it connects. The ordered edges continue to be selected and added to the network structure only if the resulting network can be embedded into a plane i.e. can be drawn on a planar surface without edges crossing. (There are some tests for planarity based on Kuratowski's Theorem. For more details on these refer to Hopcroft and Tarjan [12].) The algorithm ends when all vertices $v_1, v_2, \ldots, v_n$ are connected, using $3(n-2)$ edges.

Mantegna [6] illustrates how the MST can extract the hierarchical arrangement between the vertices of a correlation network (in particular between stocks). Similar to this, one of the key properties of the AG, threshold networks and PMFG is that cliques can form between the vertices in the network which can highlight relationships. Huang et al. [13] create threshold networks to analyse the Chinese stock market using a correlation threshold value $-1 \leq \theta \leq 1$ where θ is the correlation coefficient between two stocks. They study the relationship between the maximum clique, maximum independent set (a subset $I \subseteq V$ such that the subgraph $G(I)$ has no edges) and the threshold value θ. Huang et al. [13] state that '*the financial interpretation of the clique in the stock correlation network is that it defines the set of stocks whose price fluctuations exhibit a similar behaviour*'.

The PMFG has proven to be an important tool for filtering the most relevant information from a network, particularly in correlation based networks that model the correlation between stock prices. For example, Pozzi et al. [14] consider the level of risk and the returns on portfolios selected using filtered graphs, including PMFG. Erygit et al. [15] use PMFGs (along with MSTs and clustering methods) to analyse the daily and weekly return correlations among indices from stock exchange markets of 59 countries. In general, the PMFG can tell us about the clusters (as mentioned above) that form within the dataset, regardless of the network nature, as a result of the underlying topological properties of the network. Song et al. [16] introduce a technique to extract the cluster structure and detect the hierarchical organization within a complex dataset. This method has been developed using the topological structure of the PMFG such as the separating 3-cliques which separate a graph into two disconnected parts. Aste et al. [17] discuss the benefits of studying networks in terms of their surface embeddings. By considering the topology of the PMFG we can see that the basic structure (or motif) of the PMFG is a series of 3-cliques. Consider a sphere, a surface with $g = 0$. The PMFG separates the sphere into a sequence of triangular faces, with each vertex of the network belonging to a 3-clique. For a set of vertices there are various representations that this underlying series of 3-cliques can form (see Section 2.2). A set of three 3-cliques joined by the shared edges of a fourth 3-clique will form a 4-clique between a group of four vertices. Aste et al. [18] discuss that there must be strong relations between the properties of these 4-cliques and the ones of the system from which the cliques have been generated. For the PMFG we consider the vertices that form the 3- and 4-cliques (as the maximum number of elements that can form a clique is 4). Tumminello et al. [11] state '… *normalizing quantities are $n_s - 3$ for 4-cliques and $3n_s - 8$ for 3-cliques. Although we lack a formal proof, our investigations suggest that these numbers are the maximal number of 4-cliques and 3-cliques, respectively, that can be observed in a PMFG of n_s elements*'. As well as looking at the average correlations within the cliques and whether the cliques are from one sector or cross-sector we also consider the ratio between the number of cliques that have formed to the maximum number of cliques that could form. For this, Tumminello [11] used the normalizing quantities that have been mentioned above, an approach that has also been used in Refs. [15,18,19] and used when defining the connection strength of a sector in Ref. [20]. This paper provides the missing formal proof that $3n - 8$ and $n - 4$ are indeed the maximum number of 3-cliques and 4-cliques possible in a PMFG.

This paper is organized as follows: Section 2 provides relational definitions and notations including diagonal flips (2.1) and surface triangles and separating 3-cycles (2.2). Section 3 discusses various representations of each maximal planar graph, including standard spherical triangulation form (3.1). Our main results are presented in Section 4. Section 5 concludes and gives future direction for our research.

2. Relational definitions and notations

Here we introduce some key terminology that we use throughout the paper. Consider a graph $G(V, E)$ where V is the set of vertices belonging to G and E is the set of edges belonging to G. We denote the number of vertices $|V| = n$ and the number of edges $|E| = e$.

Let G be a *planar graph*, i.e. a graph that can be embedded in the plane in such a way that the edges of G will only intersect at the end points (the vertices of G). The planar graph divides the plane into *faces*, with each face bound by a simple cycle of G. The number of edges in this boundary is the *degree* of the face. The *planar representations* of G are all possible isomorphic embeddings of G in the plane.

A *triangulation* of a closed surface is a simple graph, one that does not contain self- or multiple-edges, which is embedded into the surface so that each face is a triangle and that two faces meet along at most one edge. A planar graph is *maximal* if it is triangulated because if a face has more than 3 edges we can add a diagonal edge. A PMFG is a triangulation of a sphere. Within this paper we shall denote P_n as a *maximal planar graph* with n vertices.

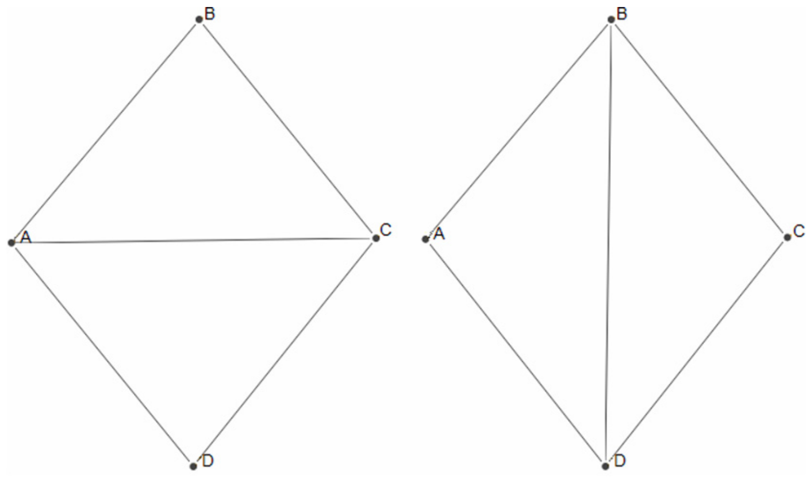

Fig. 1. A quadrilateral ABCD is formed by the two adjoining triangles ABC and ACD which share a common edge (A, C). If we perform a diagonal flip the edge (A, C) is replaced by the edge (B, D).

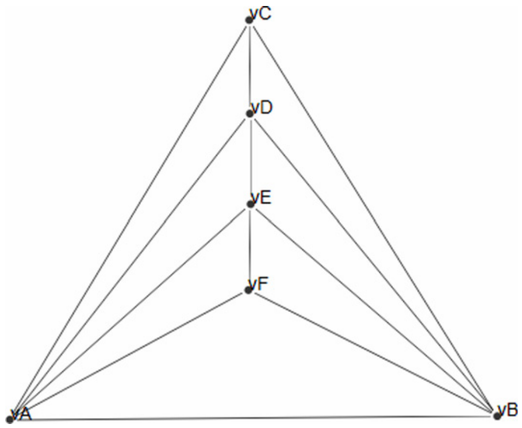

Fig. 2. This PMFG with 6 vertices highlights the two possible 3-cliques. Vertices A, C, D form a 3-clique and they outline a triangle on the surface. Vertices A, B, E also form a 3-clique however they do not outline a surface triangle but rather the edges enclose 3 surface triangles which share common edges. A, B, E form a separating 3-cycle.

A *chord* is an edge connecting two vertices of a cycle, which is not included in the cycle itself. For a graph P_n, a cycle of length k ($k \geq 3$) is called a k-cycle, denoted as $\mathcal{C}_k$. A cycle $\mathcal{C}$ is a *pure chord-cycle* if the interior of $\mathcal{C}$ contains no vertices and all the interior faces of $\mathcal{C}$ are triangles. If each of the cycles of 4 or more vertices within a graph has a chord, then the graph is called a *chordal graph*. A *wheel graph*, denoted as W_n, is a graph with $n \geq 4$ formed by connecting a single vertex to all other vertices of an $(n-1)$-cycle.

2.1. Diagonal flips

Consider two triangular faces which share a common edge and form a quadrilateral, (see Fig. 1). Negami [21] defines a *diagonal flip* of an edge as replacing the existing common edge with a new edge between the other two vertices. A diagonal flip is only possible if the resulting quadrilateral does not contain any multiple edges.

2.2. Surface triangles and separating 3-cycles

For a graph $G(V, E)$ a subset of vertices $C \subseteq V$ is called a *clique* if the subgraph $G(C)$ is a complete graph and is denoted as C_m where $|C| = m$. As a result of Kuratowski's Theorem we know that the PMFG allows cliques up to a maximum size of 4 vertices and so in this paper we will consider cliques of sizes 3 and 4. The 3-cliques can take the form of triangles on the surface (a pure chord-cycle of length 3 that forms a face of the PMFG) or a *separating 3-cycle* (a 3-cycle where both the interior and exterior of C_3 contain one or more vertices), (see Fig. 2).

The faces bounded by a cycle of edges are called *finite faces* whereas the unbounded face (ABC in Fig. 2) is called the *infinite face*. As the PMFG is a triangulation of the sphere this unbounded infinite face will also form a triangle.

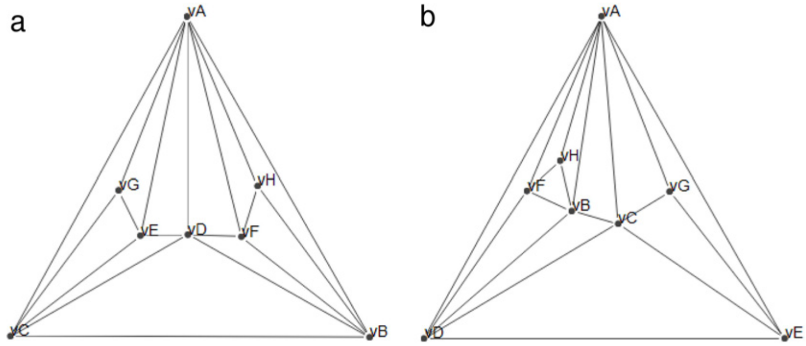

Fig. 3. PMFGs with $n = 8$ and $\deg[v_A, v_B, v_C, \ldots, v_H] = [7, 5, 5, 5, 4, 4, 3, 3]$. These graphs are isomorphic and have $C_3 = 16$ and $C_4 = 5$.

In Section 4 we will study the maximum number of 3-cliques possible in a PMFG, however we begin by studying the maximum number of triangles on the plane (i.e. the maximum number of faces). To do this we use the Handshaking Lemma and Euler's formula (refer to Ref. [22] for further details). For the remaining of this section let $G(V, E)$ be a simple, undirected, finite planar graph.

Proposition 1. *Let G be a PMFG with n vertices, e edges and f faces. Then $e = 3n - 6$ and $f = 2n - 4$.*

Proof. Since G is planar and $\deg(v_i) \geq 2$,

$$\sum_{i=1}^{f} \deg(f_i) = 2e.$$

Since G is a triangulation, $\deg(f_i) = 3 \Rightarrow 3f = 2e$.
We can substitute into Euler's formula and obtain the following,

$$n - \frac{3}{2}f + f = 2$$

$$f = 2n - 4 \tag{1}$$

and similarly,

$$n - e + \frac{2}{3}e = 2$$

$$e = 3n - 6. \quad \square \tag{2}$$

So for a PMFG, the triangulation of a sphere, we have a maximum number of $2n - 4$ surface triangles.

3. Representations of each maximal planar graph

In this section the various representations of planar graphs are presented, which are used to achieve the main results shown in Section 4.

Using the relationship between the number of edges and the sum of the vertex degree we can calculate the maximum sum of all vertex degrees for a PMFG with n vertices. By considering all combinations of the possible degrees of each vertex we can see what embeddings would be possible and, from these, which would be planar graphs. The following worked example shows this more clearly.

Example 1. Take $G(V, E)$ where $|V| = 8$ then we know:

$$e = 3n - 6 = 18 \quad \text{and so} \quad \sum_{i=1}^{8} \deg(v_i) = 2e = 36.$$

Then each vertex can have a degree value from the set $\{3, 4, 5, 6, 7\}$ due to the restrictions that each vertex can be joined to all other vertices at most once (the PMFG does not allow for multiple edges) and the degree of each vertex must be at least 3. From this set there are 53 possible combinations that would give the total degree sum of 36 and from these combinations 13 would produce a planar graph. Graphs that have the same combination of vertex degrees and are isomorphic to each other are known as *planar representations* and they will have the same number of 3-cliques (denoted C_3) and 4-cliques (denoted C_4), (see Fig. 3).

It is possible however to have graph structures with the same combination of vertex degrees that are not isomorphic, (see Fig. 4).

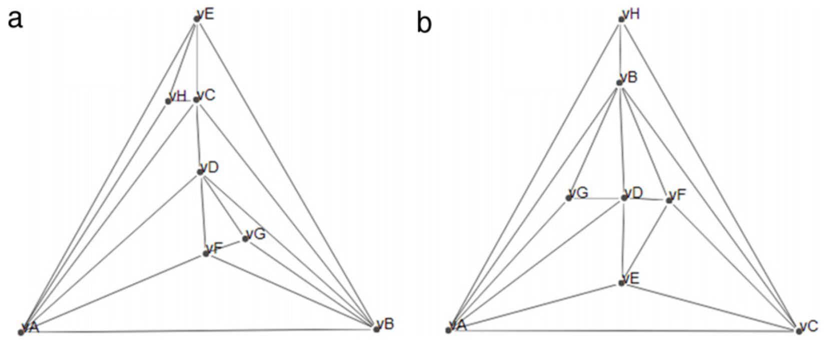

Fig. 4. Both graphs have $\deg[v_A, v_B, v_C, \ldots, v_H] = [6, 6, 5, 5, 4, 4, 3, 3]$ however they are not isomorphic and so are not planar representations of a single graph. Furthermore, they have different numbers of C_3 and C_4 with the graph shown in Panel (a) having $C_3 = 16$, $C_4 = 5$ and the graph shown in Panel (b) having $C_3 = 14$, $C_4 = 2$.

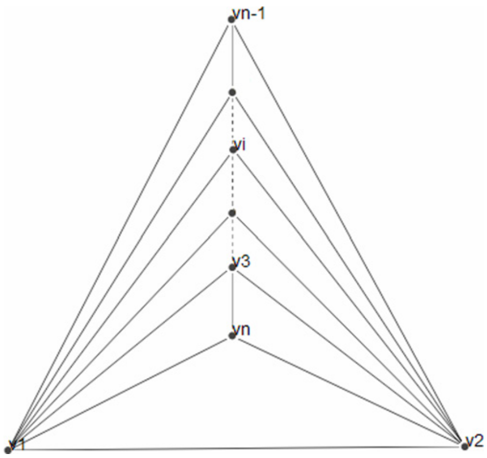

Fig. 5. A maximal graph with n vertices in the standard spherical triangulation.

3.1. Standard spherical triangulation form

We now consider how these different embeddings relate to each other using the idea of diagonal flips, as introduced by Negami [21].

In 1936 Wagner [23] proved that any two triangulations of the sphere can be transformed into each other by a finite series of diagonal flips (see also Ref. [24]). Although this does not hold for surfaces in general it has been shown to be true for triangulations of the torus, projective plane and Klein bottle. Negami [21] generalized Wagner's result:

Theorem 1. *For any closed surface F^2, there exists a positive integer N such that two triangulations G and G' of F^2 are equivalent to each other under diagonal flips if $|V(G)| = |V(G')| \geq N$.*

In the case of the PMFG, the triangulation of the sphere, $N = 4$.

Lemma 1. *Any maximal graph with n vertices, $n \geq 4$, can be transformed to the standard spherical triangulation (or normal form), (see Fig. 5), using a series of diagonal flips.*

(For proof refer to Ref. [25].)

From $n \geq 4$, the degrees of each vertex in the standard spherical triangulation are as follows:

$$\deg[v_1, v_2, v_3, v_4, \ldots, v_i, \ldots, v_{n-1}, v_n] = [n-1, n-1, 4, \ldots, 4, \ldots, 3, 3].$$

4. Main results

4.1. Generating maximal planar graphs

In 1891 Eberhard proposed a system in which a combination of a set of three operations could generate all possible maximally (filtered) planar graphs. We begin with the complete graph K_4 and then choose a generating operation, $\varphi_1, \varphi_2, \varphi_3$, from the operation set, Φ. Each generating operation adds a new vertex to the graph. This system is denoted as $\langle K_4; \Phi = \{\varphi_1, \varphi_2, \varphi_3\}\rangle$.

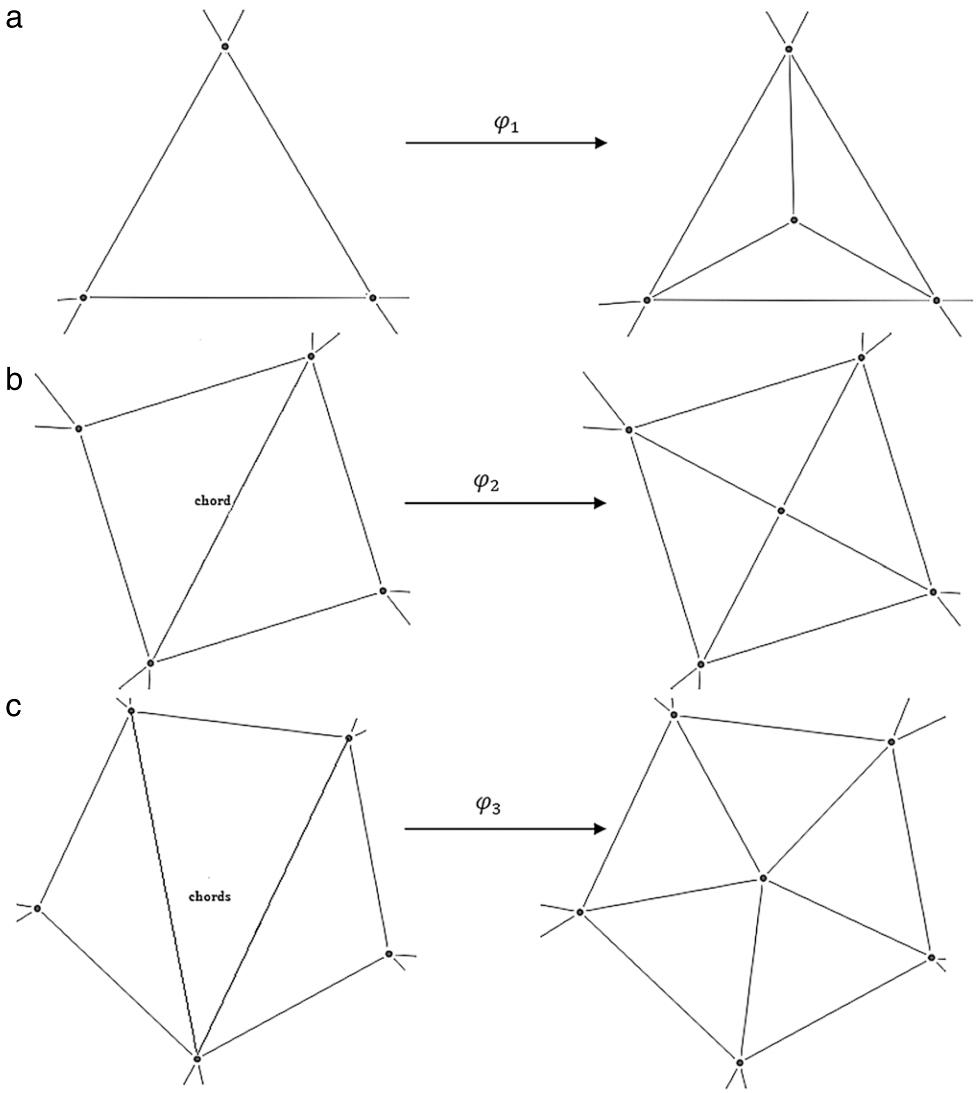

Fig. 6. Panel (a)—The first Eberhard operation, φ_1, Panel (b)—The second Eberhard operation, φ_2, Panel (c)—The third Eberhard operation, φ_3.

For $\varphi_1, \varphi_2, \varphi_3$ begin by deleting all of the chords of a pure chord-cycle $\mathcal{C}_k$ with length $k = (3, 4, 5)$ respectively. Then add a new vertex inside $\mathcal{C}$, which is connected to all vertices of $\mathcal{C}$ so that a wheel subgraph is created.

Fig. 6 shows how a pure chord-cycle transforms under each Eberhard operation.

Example 2. For P_6 two of the five possible representations are planar. Both of these graphs (shown in Fig. 7) can be generated using a series of Eberhard's operations, starting with K_4.

We first generate P_5 from the complete graph K_4. There is only one representation of P_5 that is planar.

Then we can generate P_6 from P_5, (shown in Fig. 8), by using any of the three operations from Φ. In P_5 there are five pure chord-cycles of length 3 which are as follows: (A, B, E), (B, D, E), (B, C, D), (A, C, D) and (A, D, E). For φ_1 we add a vertex to the interior of one of the above pure chord-cycles which we join to the three edges of the cycle to create a wheel subgraph and a representation of P_6. Note that all $\mathcal{C}_3$ in P_5 will generate P_6 in the standard spherical triangulation form, (see Fig. 9).

In P_5 there are four pure chord-cycles of length 4, all with one chord edge, which are as follows: (A, B, C, D), (A, B, D, E), (A, C, D, E) and (B, C, E, D). For φ_2 we select one of these pure chord-cycles and start by deleting the chord edge. Then we add a vertex to the interior which we join to the four edges of the cycle to create a wheel subgraph and a representation of P_6. Using φ_2 can generate P_6 in both standard spherical triangulation form and the alternative form, depending on which pure chord-cycle is chosen, (see Fig. 10).

The final option is to use φ_3 and the one pure chord-cycle of length 5 (A, C, B, D, E). This has two chord edges (A, D) and (C, D) which will be removed before adding a new vertex to the interior. This is then joined to each of the five vertices in the cycle to produce P_6 in standard spherical triangulation form.

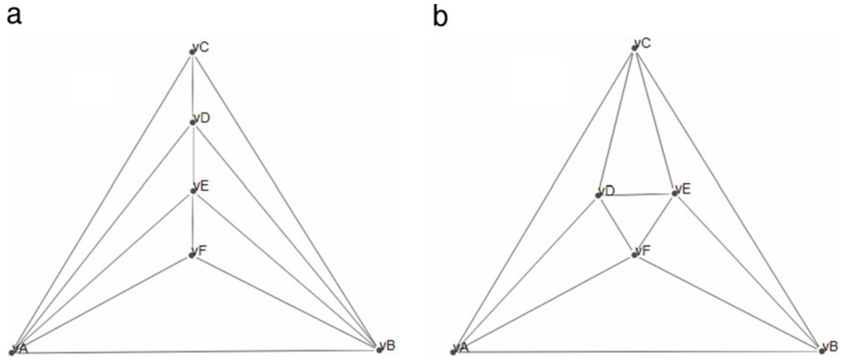

Fig. 7. Panel (a) Standard spherical triangulation form, with $C_3 = 10$, $C_4 = 3$ and Panel (b) the alternative form, with $C_3 = 8$, $C_4 = 0$.

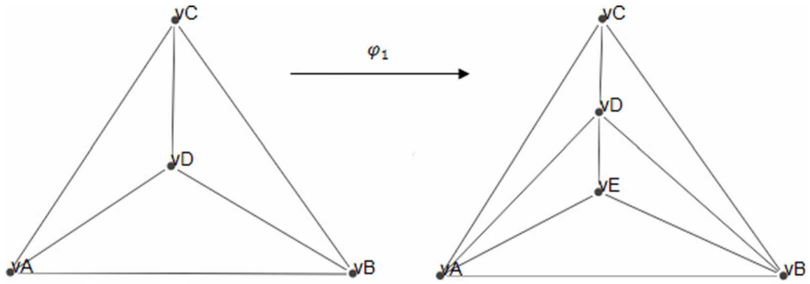

Fig. 8. The transformation of K_4 to P_5 using Eberhard operation φ_1.

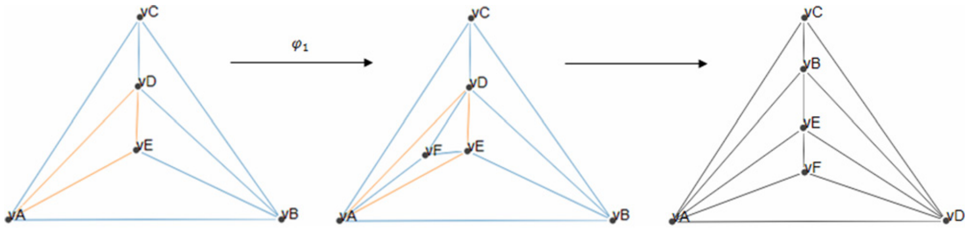

Fig. 9. The transformation of P_5 to P_6 using Eberhard operation φ_1.

4.2. Maximum number of 3- and 4-cliques

Using the Eberhard operations to generate maximal planar graphs we can find the maximum number of 3-cliques that will be added during each iteration of the construction algorithm and so consequently the maximum number of 3-cliques that is possible in P_n.

Now the maximum number of 3-cliques is shown using Theorem 2.

Theorem 2. *Let P_n be a maximal planar graph with n vertices, $n \geq 3$. Then the maximum number of 3-cliques, $C_3^{\max}$, that are possible is $C_3^{\max}(P_n) = 3n - 8$.*

Proof. With each Eberhard operation there is a new vertex added and also a certain number of 3-cliques—φ_1 adds three new 3-cliques whereas φ_2 and φ_3 both add two new 3-cliques.

Therefore the maximum number of 3-cliques that can be added is three and so we can say that:

$$C_3^{\max}(P_n) \leq C_3(K_4) + 3(n-4), \quad (3)$$

where $(n - 4)$ is the number of Eberhard operations required to construct P_n. As we know that the number of 3-cliques in K_4 is always 4 we can simplify (3) and obtain:

$$C_3^{\max}(P_n) \leq 4 + 3n - 12 = 3n - 8.$$

So the maximum number of 3-cliques possible in P_n, $C_3^{\max} = 3n - 8$. □

We can apply a similar argument to obtain the maximum number of 4-cliques in P_n.

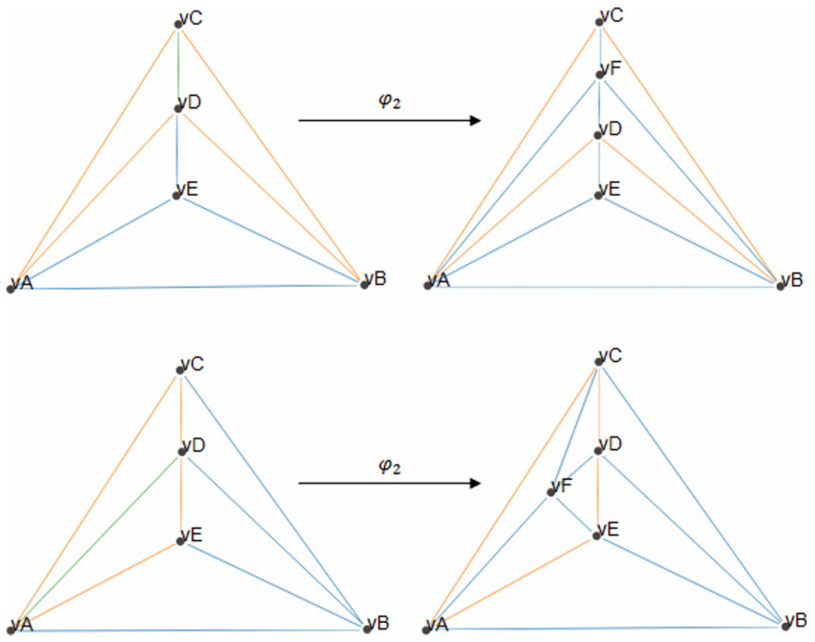

Fig. 10. The transformation of P_5 to P_6 using Eberhard operation φ_2.

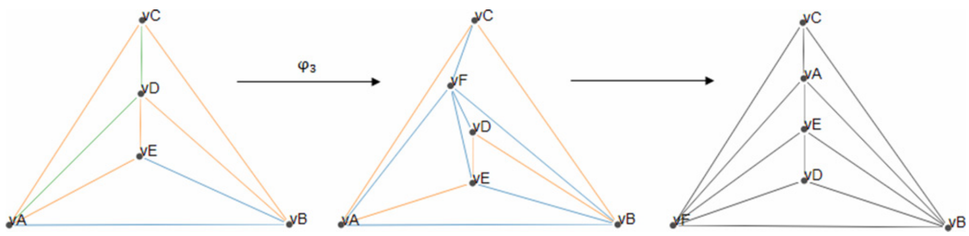

Fig. 11. The transformation of P_5 to P_6 using Eberhard operation φ_3.

Theorem 3. *Let P_n be a maximal planar graph with n vertices, $n \geq 3$. Then the maximum number of 4-cliques, $C_4^{\max}$, that are possible is $C_4^{\max} = n - 3$.*

Proof. With each Eberhard operation there is a new vertex added, however only φ_1 adds one new 4-clique, neither φ_2 or φ_3 adds any new 4-cliques.

Therefore the maximum number of 4-cliques that can be added is one and so we can say that:

$$C_4^{\max}(P_n) \leq C_4(K_4) + (n - 4), \tag{4}$$

where $(n - 4)$ is the number of Eberhard operations required to construct P_n. As we know that the number of 4-cliques in K_4 is always 1 we can simplify (4) and obtain:

$$C_4^{\max}(P_n) \leq 1 + n - 4 = n - 3.$$

So the maximum number of 4-cliques possible in P_n, $C_4^{\max} = n - 3$. □

4.3. 3- and 4-cliques in the standard spherical triangulation form

As discussed in Section 3 there can be various representations of a graph and the number of 3-cliques that forms between the vertices will depend on the structure of the graph. The minimum number of 3-cliques that will form in a PMFG with n vertices $= 2n - 4$ as it will be equal to the number of surface triangles, including the vertices that form the infinite face (as shown in Proposition 1). From 4.2 we now have an expression for the maximum number of 3-cliques that can form in a PMFG. We now show that the standard spherical triangulation form always contains the maximum number of 3-cliques. When a maximal planar graph is in the standard spherical triangulation form we have two types of 3-cliques—firstly those formed by surface triangles and secondly those that enclose 3 surface triangles which share common edges (as shown in Figs. 2 and 11) which form between a vertex and the two vertices with degree $n - 1$. We call these two forms of triangles surface triangles and enclosing triangles respectively.

Theorem 4. *For a maximal graph with n vertices in the standard spherical triangulation form the number of* $C_3 = 3n - 8$ *and the number of* $C_4 = n - 3$.

Proof. Number of C_3 in the standard spherical triangulation form = (the number of surface triangles) + (the number of enclosing triangles) − (unbounded face) $= (2n - 4) + (n - 3) - 1 = 3n - 8$.

Note that the number of enclosing triangles is equal to what we have shown to be the maximum number of C_4. □

5. Conclusion

In this paper we have discussed possible embeddings of n-vertex triangulations in the form of maximal planar graphs. We used the generating operations proposed by Eberhard to construct these maximal planar graphs and have proven that the maximum number of 3-cliques that can exist in a maximal planar graph with n vertices is $3n - 8$ and the maximum number of 4-cliques that can exist is $n - 3$, where the number of vertices $n \geq 4$. This is true for when a maximal planar graph is constructed using the PMFG algorithm.

Finally, we have shown how any maximal planar graph can be transformed to a standard spherical triangulation form retaining the original number of vertices and edges and that this structure will always contain the maximum number of 3- and 4-cliques.

The work on PMFG has recently been extended to include Partial Correlation Planar maximally filtered Graphs (PCPG) which use partial correlations between the stocks as opposed to Pearson correlations [26,27]. It would be interesting to see how the use of partial correlations affects the 3- and 4-cliques. We leave this for future work.

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