Ph225b   **Problem Set #2 (Chapter IV.3)**   January 31, 2005
(due on February 14, 2005)

1. Magnetic monopoles and Dirac quantization

In our discussions of the topological object in (3+1)-dimensional space-time, we introduced a tensor field

$$\mathcal{F}_{\mu\nu} \equiv \frac{F^a_{\mu\nu}\varphi^a}{|\varphi|} - \frac{\varepsilon^{abc}\varphi^a \left(D_\mu \varphi\right)^b \left(D_\nu \varphi\right)^c}{e|\varphi|^3}, \qquad \text{(P2.1)}$$

where $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu$, $D_i\varphi^a = \partial_i\varphi^a + e\varepsilon^{abc}A_i^b\varphi^c$, and the gauge potential A_i^b satisfies the following condition at spatial infinity:

$$\lim_{r\to\infty} A_i^b \to \frac{1}{e}\varepsilon^{bij}\frac{x^j}{r^2}. \qquad \text{(P2.2)}$$

(a) Show that within a spatial region where φ^a is a constant, the tensor field $\mathcal{F}_{\mu\nu}$ as defined above indeed represents the electromagnetic field strength.

(b) Find the magnetic field $\vec{B}$ from the above tensor field in the limit of $r \to \infty$, and show that the total flux from a magnetic monopole with $\vec{B}$ thus defined satisfies the Dirac quantization condition.

2. The Hopf non-local Lagrangian and the mass dimension of the Chern-Simons term

Consider the following Lagrangian in (2+1)-dimensional space-time that involves the Chern-Simons term $\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$ and a Lagrangian L_0 with a conserved current j^μ:

$$L = L_0 - \gamma\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + a_\mu j^\mu, \qquad \text{(P2.3)}$$

where a_μ denotes a gauge potential.

(a) Show that with the choice of the Lorentz gauge $\partial_\mu a^\mu = 0$, one can integrate out the gauge potential a in Eq. (P2.3) and obtain the following non-local Lagrangian:

$$L_{Hopf} = \frac{1}{4\gamma}\left(j_\mu \frac{\varepsilon^{\mu\nu\lambda}\partial_\nu}{\partial^2} j_\lambda \right). \qquad \text{(P2.4)}$$

The Lagrangian L_{Hopf} in Eq. (P2.4) is known as the Hopf term, which, in the fractional quantum Hall fluids, is related to the quasiparticle interactions.

(b) Consider the mass dimensions of the Chern-Simons term and the Maxwell term in (2+1)-dimensional space-time. Prove that at long distances the Maxwell term becomes irrelevant relative to the Chern-Simons term.