Ph225a **Problem Set #5 (Chapter II.7. & II.8.)** November 17, 2004
(due date: December 1, 2004)

1. The Lagrangian of the Yukawa theory is given by:

$$L = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m_p\right)\psi + \frac{1}{2}\left[\left(\partial\varphi\right)^2 - \mu_p{}^2\varphi^2\right] - \lambda_p\varphi^4 + f_p\varphi\bar{\psi}\psi + \text{(counter terms)},$$

where all terms are consistent with the definitions in the class notes. The Yukawa theory is known to be renormalizable in 4-dimensional space-time with a superficial degree of divergence D of a diagram only dependent on the external bosonic (B_E) and external fermion lines (F_E):

$$D = 4 - B_E - \frac{3}{2}F_E\,. \tag{1}$$

The above result can be obtained from the following relations:

$$V_f + 4V_\lambda = B_E + 2B_I\,; \qquad 2V_f = F_E + 2F_I\,;$$
$$L = B_I + F_I - (V_f + V_\lambda - 1); \qquad D = 4L - 2B_I - F_I\,, \tag{2}$$

where B_I and F_I represent the internal bosonic and fermionic lines, respectively, L is the number of loops in the diagram, V_λ denote the number of vertices with bosonic coupling λ to φ in a diagram, and V_f is the number of vertices with bosonic coupling f to ψ in a diagram. Explain how the four relations in Eq. (2) that lead to Eq. (1) are obtained.

2. (a) In our derivation of the effective potential $V_{\text{eff}}(\varphi)$ of a scalar field φ under spontaneous symmetry breaking, we have defined an effective action $\Gamma(\varphi_c)$ in terms of the connected generating functional $W(J)$ in the presence of the source $J(x)$ as:

$$\Gamma(\varphi_c) = W(J) - \int d^4x\, J(x)\,\varphi_c(x)\,,$$

where φ_c represents the classical field as defined in the class notes. Show that the functional $\Gamma(\varphi_c)$ can be expanded in the following form:

$$\Gamma(\varphi_c) = \int d^4x\left[-V_{eff}(\varphi_c) + \frac{1}{2}\left(\partial\varphi_c\right)^2 Z(\varphi_c) + \cdots\right],$$

where Z is a function of φ, and $V_{\text{eff}}(\varphi)$ is the effective potential that incorporates quantum fluctuations. This method of Jona-Lasinio enables us to obtain the true vacuum state of the system by finding the minima of the effective potential $V_{\text{eff}}(\varphi)$, which is in contrast to the semi-classical approximation that searches for minima of the potential $V(\varphi)$ given in the Lagrangian.

(b) The Coleman-Weinberg effective potential derived in the class notes is reproduced below:

$$V_{eff}\left(\varphi_c\right) = V\left(\varphi_c\right) - \frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2 - V''(\varphi_c)}{k^2}\right] + B\varphi_c{}^2 + C\varphi_c{}^4 + O\left(\hbar^2\right).$$

Following the discussions in the notes, verify that the coefficients B and C associated with the counter terms are given by:

$$B = -\frac{\Lambda^2}{64\pi^2}\lambda, \qquad C = \frac{\lambda^2}{(16\pi)^2}\left[\log\left(\frac{2\Lambda^2}{\lambda\langle\varphi_c\rangle^2}\right) - \frac{11}{3}\right],$$

where Λ is the ultraviolet cutoff, $\langle\varphi_c\rangle$ is the vacuum expectation value upon spontaneous symmetry breaking under quantum fluctuations, and λ is the bare coupling constant defined in the potential

$$V\left(\varphi\right) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4\,.$$