Ph225a **<u>Problem Set #3</u> (Chapter II.5.)** October 20, 2004
(due date: November 1, 2004)

1. In our discussion of the Weyl's spinors for the Dirac field, we have written the effective Dirac equation in the rest frame of the fermions:

$$\left(\gamma^0 - 1\right)\Psi(p_r) = 0\,,$$

where $\Psi(p_r)$ is the rest frame Dirac spinor. We have mentioned that by performing a Lorentz boost $\exp\left(-i\,\vec{\theta}\vec{K}\right)$, where $\vec{K}$ is the Lorentz boost defined in the class notes and $\vec{\theta}$ is the angle of Lorentz transformation (such that the momentum can be written as $\vec{p} = m\,\hat{\theta}\sinh|\vec{\theta}\,|$), we can derive the Dirac equation from the rest frame so that

$$\left(\frac{\gamma^\mu p_\mu}{m} - 1\right)\Psi(p) = 0\,,$$

provided that $e^{-i\vec{\theta}\vec{K}}\gamma^0 e^{i\vec{\theta}\vec{K}} = \left(\gamma^\mu p_\mu / m\right)$. Prove the last relation explicitly.

2. Verify the following useful identities associated with the γ-matrices:

(a) $\gamma^\rho \gamma^\mu \gamma_\rho = -2\,\gamma^\mu$.

(b) $\gamma^\rho \gamma^\mu \gamma^\nu \gamma_\rho = 4\eta^{\mu\nu}$. ($\eta^{\mu\nu}$: Minkowski metric)

(c) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(\eta^{\mu\nu}\,\eta^{\rho\sigma} - \eta^{\mu\rho}\,\eta^{\nu\sigma} + \eta^{\mu\sigma}\,\eta^{\nu\rho})$

3. (a) Show that the bilinear $\bar{\Psi}\gamma^\mu\Psi$ transform under the Lorentz group and parity as a vector, and $\bar{\Psi}\sigma^{\mu\nu}\Psi$ transforms like a tensor. Here Ψ denotes the Dirac spinor, γ^μ the γ-matrices, and $\sigma^{\mu\nu} = i\,[\gamma^\mu, \gamma^\nu]/2$.

(b) Prove that the $SO(3,1)$ spinor representation (½, ½) is a Lorentz vector. [*Hint*: You may first work out the explicit representation following similar lines of discussions given in the notes for the representations (½, 0) and (0, ½), and then consider the transformation law under infinitesimal Lorentz transformation.]

4. In our canonical quantization of the Dirac field, we decompose $\Psi(x)$ into plane waves, with u spinors corresponding to positive energy particles (moving forward in time) and v spinors corresponding to negative energy particles (moving backward in time). In the rest frame we have the spinors given by

$$u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad v_1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Now consider a pure Lorentz boost with a momentum $\vec{p} = (0, 0, p_z)$. If the rest mass of each particle under consideration is m, find the Lorentz transformed u and v spinors by applying the corresponding Lorentz transformation matrix $S(\Lambda)$ to the independent spinor basis in the rest frame.