July 28 Workshop - Markov Processes and Stochastic Matrices

Question 1. Consider the (2×2)-matrix A given by

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Our goal is to find a closed form for A^n.

(a) Find the characteristic polynomial and the eigenvalues of A.

(b) Find the corresponding eigenspaces to the eigenvalues you found in (a).

(c) If A is similar to a diagonal matrix D, what are its diagonal elements?

(d) Use (b) to find the conjugating matrix P that gives $P^{-1}AP = D$.

(e) Compute P^{-1}.

(f) Verify that indeed $P^{-1}AP = D$.

(g) Use the above to find a closed form for A^n.

Question 2. Recall that the Perron-Frobenius Theorem stipulates that any stochastic matrix A has a steady-state $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$ by proving that A has 1 as an eigenvalue. Furthermore, any other eigenvalue λ for A will have $|\lambda| \leq 1$.

(a) If we are given a system with two possible states, then our stochastic matrix will be a (2×2)-matrix. Argue that this stochastic matrix will be of the form

$$A = \begin{bmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{bmatrix}$$

for probabilities $0 \leq \alpha, \beta \leq 1$.

(b) Compute the eigenvalues of the stochastic matrix A using the parameters α and β. Verify that, as guaranteed by the Perron-Frobenius Theorem, one of the eigenvalues of A is 1.

(c) Verify that the eigenvalue λ you found in (b) distinct from 1 has $|\lambda| \leq 1$. For our (2×2)-case, why must it always be true that this other eigenvalue is real?

(d) Compute a steady state vector $\mathbf{x}$ by finding the eigenspace corresponding to the eigenvalue $\lambda = 1$.

Question 3. Choose any two linearly independent vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$ and any two real numbers λ_1 and λ_2. Use the change of basis equation to construct a (2×2)-matrix A that has $\mathbf{v_1}$ as an eigenvector with eigenvalue λ_1 and $\mathbf{v_2}$ as an eigenvector with eigenvalue λ_2.

Question 4. This questions describes a stochastic process called the *cat and mouse* game. Imagine 5 little houses lined up in a row. There is one cat and one mouse in this game, where the cat begins in house 1 and the mouse begins in house 5. Every minute, the cat and mouse are forced to simultaneously move to an adjacent house; the probability that they will move to a house on the left of them (p = .5) is equal to the probability that they will move to right of their current house (p = .5). The cat and mouse have no knowledge of the location of the other, and the game ends when the cat and the mouse are in the same house (and the cat eats the mouse).

(a) Prove that at if a cat (or mouse) is in an even-numbered box at time k, then at times $k+1$ he is in an odd-numbered house. Furthermore, if he starts off in an odd-numbered house, he must be in an even-numbered house the following minute.

(b) Use (a) to show that at every time step, the cat and the mouse are both in even-numbered houses or both in odd-numbered houses.

(c) Use (b) to show that there are exactly five states to our cat and mouse system:

   (i) State 1 - Cat in house 1, Mouse in house 5

   (ii) State 2 - Cat in house 1, Mouse in house 3

   (iii) State 3 - Cat in house 2, Mouse in house 4

   (iv) State 4 - Cat in house 3, Mouse in house 5

   (v) State 5 - The game terminates (Cat and Mouse in same house)

(d) Compute the stochastic matrix A associated to this system by finding p_{ij}, the probability that the system will transition from state i to state j.

(e) Verify that your matrix A is indeed stochastic by showing that the columns sum to 1.

(f) Verify that our system begins in state 1 and thus our starting matrix is $\mathbf{x_0} = [1, 0, 0, 0, 0]$ (because we know with probability 1 that we are in state 1 and with probability 0 that we are in the other states at the beginning of the game.

(g) Compute $\mathbf{x_k} = A^k \mathbf{x_0}$ for $k = 1, 2, 3, 4$. The i-th component of the vector $\mathbf{x_k}$ will give the probability that at time k, the system is in that state.

(h) What do you think the game will tend towards as k get large?

(i) Describe how you would find exactly what the game will tend towards (you don't have to actually compute this, but simply give the steps involved).