July 27 Workshop - Dimensions, Determinants, and Eigenspaces

Question 1. Consider the set

$$SL_n(\mathbb{R}) = \{A \mid \det(A) = 1\},$$

where A is an $n \times n$ matrix. Show that $SL_n(\mathbb{R})$ is a *group* with the operation of matrix multiplication by proving the following.

(a) Closed under Multiplication. If $A, B \in SL_n(\mathbb{R})$, then $A \cdot B \in SL_n(\mathbb{R})$.

(b) Existence of Identity. Prove that there exists some element $I \in SL_n(\mathbb{R})$ such that $I \cdot A = A \cdot I = A$ for all $A \in SL_n(\mathbb{R})$.

(c) Closed under Inverses. Prove that for every $A \in SL_n(\mathbb{R})$, there exists some $A^{-1} \in SL_n(\mathbb{R})$ such that $A \cdot A^{-1} = I_n$.

Question 2. A square matrix A is *similar* to a square matrix B if there is some invertible matrix P such that

$$A = PBP^{-1}.$$

These similarities are very important in changing basis.

(a) If A and B are similar, show that $\det(A) = \det(B)$.

(b) Use (a) to show that A is invertible if and only if B is invertible.

(c) Show that if A is invertible (and thus B is also invertible), then A^{-1} is similar to B^{-1}.

(d) If A and B are similar, show that they have the same characteristic polynomial and thus have equivalent eigenvalues.

(e) Find the the eigenvalues of the (2×2)-matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Use this and (d) to show that A is not similar to the identity matrix.

Question 3. Consider the (3×3)-matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

(a) Find the characteristic polynomial of A. Use it to find the eigenvalues of A.

(b) For each of the above eigenvalues, compute the corresponding eigenspaces.

(c) Sketch the eigenspaces in $\mathbb{R}^3$. How does the matrix A transform these eigenspaces?

Question 4. Let A be an upper triangular matrix.

(a) Show that the eigenvalues of A are precisely its n diagonal entires.

(b) Use determinants to show that A is invertible if and only if 0 is not a diagonal entry.

(c) Use (a) to show that for an upper triangular A, its determinant is equal to the product of its eigenvalues.

(d) Argue that if 0 were an eigenvalue of A, then some (non-zero) eigenvector $\mathbf{x}$ is being mapped to $\mathbf{0}$ by A; thus A is not an injective map and therefore not invertible.

(e) Use properties of determinant and transposes to show that A and A^T have the same characteristic polynomials and thus the same eigenvalues. Argue that this means that (a)-(d) hold true for lower triangular matrices as well.

Question 5. Consider the (2×2)-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We will give a complete characterization of the eigenvalues for this general (2×2)-matrix.

(a) Compute the characteristic polynomial for the matrix A in terms of the parameters a, b, c, and d.

(b) Verify that the characteristic polynomial you found in (a) is equivalent to

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A),$$

where $\mathrm{tr}(A)$ and $\det(A)$ are the trace and determinant of A, respectively.

(c) For eigenvalues to exist, the characteristic polynomial of A must have real roots. Use the characteristic polynomial in (b) to find conditions on $\det(A)$ and $\mathrm{tr}(A)$ that will ensure that real roots (and thus eigenvalues) exist.

(d) Under what conditions on $\mathrm{tr}(A)$ and $\det(A)$ will we have a single eigenvalue λ with multiplicity 2?