July 26 Workshop - Subspaces, Vector Spaces, and Bases

Question 1. Let $\mathbb{P}_n$ be the space of polynomials of degree at most n with real coefficients. Of course, $\mathbb{P}_n$ has a natural additive structure (addition of polynomials) and scalar multiplicative structure (multiply each coefficient of the polynomial by our real scalar). In what follows, we will prove that $\mathbb{P}_n$ is a vector space and an important linear map on $\mathbb{P}_n$.

(a) Prove that if $f, g \in \mathbb{P}_n$, then $\alpha f \in \mathbb{P}_n$ and $f + g \in \mathbb{P}_n$.

(b) What is the additive identity in $\mathbb{P}_n$ and what is the additive inverse of $f \in \mathbb{P}_n$?

(c) Show that the vectors $1, x, x^2, \ldots, x^n$ form a basis for $\mathbb{P}_n$ and thus $\dim(\mathbb{P}_n) = n + 1$.

(d) Let $D : \mathbb{P}_n \to \mathbb{P}_n$ be the map that sends $f(x)$ to $f'(x)$. Show that D is a linear map; that is, show that if $a \in \mathbb{R}$ and $f, g \in \mathbb{P}_n$, then $D(a \cdot f) = a \cdot D(f)$ and $D(f + g) = D(f) + D(g)$.

(e) Give a basis for the kernel and the image of D; what are their dimensions?

Question 2. Recall that if A is an $(n \times m)$-matrix, then it may be viewed as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $\mathbf{x} \mapsto A\mathbf{x}$.

(a) Show that $\mathrm{Ker}(A)$, the kernel of A, always contains the zero vector $\mathbf{0}$.

(b) Show that $\mathrm{Ker}(A) = \mathbf{0}$ if and only if the map given by A is one-to-one.

(c) If the columns of A are linearly dependent, show that $\mathrm{Ker}(A)$ contains a non-zero vector.

(d) Show the converse of (c). That is, show that if the columns of A are linearly independent, then $\mathrm{Ker}(A) = \mathbf{0}$.

(e) Use (b) - (d) to verify the following statement: The columns of A are linearly independent if and only if $\mathrm{Ker}(A) = \mathbf{0}$ if and only if the map given by A is one-to-one.

Question 3.

(a) Let A be an $m \times n$-matrix and let $\mathbf{e_i}$ be the n-vector with a 1 in the i-th component and 0 everywhere else. Show that $A \cdot \mathbf{e_i}$ is the i-th column of the matrix A.

(b) Use (a) to show that any $m \times n$-matrix A such that $A \cdot \mathbf{x} = \mathbf{0}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$ must be the zero matrix.

(c) Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be a basis for $\mathbb{R}^n$. Show that $\mathrm{Col}(A)$ is spanned by $A \cdot \mathbf{v_i}$.

(d) Use (c) to show that $\mathrm{Col}(A)$ is spanned by its column vectors.

(e) Use (b) to show that if $\mathrm{Col}(A) = \mathbf{0}$, then A is the zero matrix.

Question 4.

(a) Prove that any collection of n-vectors that contains the zero vector $\mathbf{0} \in \mathbb{R}^n$ cannot be a basis for $\mathbb{R}^n$.

(b) Prove that any two non-zero 2-vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$ give a basis for $\mathbb{R}^2$ if and only if $\mathbf{v_1}$ is not a scalar multiple of $\mathbf{v_2}$ (that is, there is no $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{v_2} = \mathbf{v_1}$).

(c) Find a collection of three non-zero 3-vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbb{R}^3$ that *do not* form a basis for $\mathbb{R}^3$ but no one $\mathbf{v_i}$ is a scalar multiple of any other.

Question 5. Prove that if a set of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are *linearly dependent*, then some $\mathbf{v_i}$ is in the span of the other $n-1$ vectors.

Question 6. Assume that $\mathbf{u}$ and $\mathbf{v}$ are two linearly independent n-vectors. Under what conditions on $a, b, c, d \in \mathbb{R}$ will the new vectors

$$a\mathbf{u} + b\mathbf{v} \text{ and } c\mathbf{u} + d\mathbf{v}$$

also be linearly independent?

Question 7. Let S be a collection of vectors with the property that any pair of vectors in S are linearly dependent. Show that all the vectors in S are scalar multiples of a single vector.

Question 8. Consider the following system of equations:

$$\begin{aligned} x + 4y - 3z &= 0 \\ 11x - y - 3z &= 0 \\ -5x + y + z &= 0 \end{aligned}$$

(a) Find all the solutions to the above system. What is the dimension of the solution space?

(b) What does the solution space tell us about independence of the vectors $v_1 = (1, 11, -5)$, $v_2 = (4, -1, 1)$, and $v_3 = (-3, -3, 1)$? If they are dependent, find a relation among these vectors.