July 25 Workshop - Vectors and Matrices

Question 1. The system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \; ,$$

can be represented as the matrix equation

$$A\mathbf{x} = \mathbf{b},$$

where A is the 3×3 matrix with a_{ij} as its ij-entry, $\mathbf{x}$ is the 3-vector with entries $[x_1, x_2, x_3]$, and $\mathbf{b}$ is the 3-vector with entries $[b_1, b_2, b_3]$. The vector $\mathbf{x}$ is called a solution to the system $A\mathbf{x} = \mathbf{b}$. Furthermore, a system of equations is called *homogenous* if $\mathbf{b}$ is the zero vector $\mathbf{0}$ (and thus each $b_i = 0$).

(a) Show that if $\mathbf{y}$ and $\mathbf{z}$ are two solutions to a homogenous equation $A\mathbf{x} = \mathbf{0}$, then $\mathbf{y} + \mathbf{z}$ and $\alpha\mathbf{y}$ are also solutions to the homogenous system for $\alpha \in \mathbb{R}$.

(b) Show that (a) is not necessarily true if our system is not a homogenous equation.

(c) Show that if $\mathbf{p}$ is a solution to the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$, and $\mathbf{y}$ is a solution to the corresponding homogenous equation $A\mathbf{x} = \mathbf{0}$, then $\mathbf{y} + \mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(d) Consider the system

$$\begin{aligned} 3x - 4y + 4z &= 7 \\ x - y - 2z &= 2 \\ 2x - 3y + 6z &= 5 \end{aligned} \; .$$

Write this system of equations as a matrix system $A\mathbf{x} = \mathbf{b}$. Show that $\mathbf{p} = [1, -1, 0]$ is a solution to your matrix system.

(e) Consider the *homogenous* system $A\mathbf{x} = \mathbf{0}$. Show that $\mathbf{y} = [12, 10, 1]$ is a solution to this homogenous system.

(f) Use (a) to show that $\alpha\mathbf{y}$ is also a solution to the homogenous system $A\mathbf{x} = \mathbf{0}$ for any $\alpha \in \mathbb{R}$.

(g) Use (d), (f), and (c) to find infinitely many solutions to the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

Question 2. For a square matrix A, we define the trace of A as the sum of the diagonal entries of A (and write it as $tr(A)$). Prove or disprove:
a) $tr(A + B) = tr(A) + tr(B)$
b) $tr(AB) = tr(A)tr(B)$
c) $tr(AB) = tr(BA)$.

Question 3. An $n \times n$-matrix $A = (a_{ij})$ is called *symmetric* if $a_{ij} = a_{ji}$ for all $0 \leq i, j \leq n$. Further, given an $n \times m$-matrix $B = (b_{ij})$, its transpose is the $m \times n$-matrix B^T whose i, j entry is b_{ji}. Thus, a square matrix A is symmetric if and only if $A = A^T$.

a) Let A and B be two $n \times m$ matrices and $\alpha \in \mathbb{R}$. Prove that $(A + B)^T = A^T + B^T$, $(\alpha \cdot A)^T = \alpha \cdot A^T$, and $(A^T)^T = A$

b) Prove that if A is an $n \times m$-matrix and B is an $m \times p$ matrix, then

$$(AB)^T = B^T A^T.$$

c) Show that for any $n \times n$-matrix A, the matrices $A \cdot A^T$ and $A + A^T$ are symmetric.

d) Show that for any two symmetric $n \times n$-matrices A and B, $A + B$ is symmetric

Question 4. Although matrix addition and multiplication follow many of the same rules as real addition and multiplication, there are some very important and striking differences. In particular, the existence of *zero-divisors* contrast starkly with multiplication in $\mathbb{R}$. A matrix A is said to be a zero-divisor if there exists some $B \neq \mathbf{0}$ such that $A \cdot B = \mathbf{0}$, where $\mathbf{0}$ is the matrix with 0 in every entry.

(a) Prove that the 2×2 matrix

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

is a zero-divisor.

(b) In general, prove that if a matrix is of the form

$$\begin{bmatrix} a & a \\ b & b \end{bmatrix},$$

then it is a zero-divisor.

(c) If A is a zero-divisor, prove that $B \cdot A$ is a zero-divisor for any matrix B.

(d) A matrix A is called an *idempotent* if $A^2 = A$. If A is an idempotent which is not the identity matrix, prove that A is a zero-divisor.

(e) Is is true that if A and B are both zero-divisors, then $A + B$ is also a zero-divisor? Prove or find a counterexample.

Question 5. This question regards the *Heisenberg group* $H_3(\mathbb{R})$, a subset of 3×3-matrices that is crucial in one-dimensional quantum mechanical systems. A 3×3 matrix is in the Heisenberg group if it is of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$.

(a) Show that the 3×3 identity matrix is an element of the Heisenberg group.

(b) Show that the Heisenberg group is closed under matrix multiplication. That is, show that if $A, B \in H_3(\mathbb{R})$, then $A \cdot B$ is also in $H_3(\mathbb{R})$.

(c) We can also form the *discrete Heisenberg group* $H_3(\mathbb{Z})$ by asking that the parameters a, b, and c be *integers*; that is, $a, b, c \in \mathbb{Z}$. Show that, as in (a) and (b), the discrete Heisenberg group contains the identity and is closed under matrix multiplication.