July 19 Workshop - Continuity and Differentiation

Question 1. Use induction, the product rule, and the derivative of $f(x) = x$ to show that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for every integer $n \geq 1$.

Question 2. Prove the Quotient Rule using the Product Rule and the Chain Rule.

Question 3. Use the definition of the derivative to prove that $f(x) = |x|$ is not differentiable at $x = 0$.

Question 4. Compute the derivative of $f(x) = x^x$.

Question 5. Use the definition of the derivative to compute the derivatives of $\sin x$ and $\cos x$. In your computations, you may use the following two limits without proof:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0} \frac{1-\cos h}{h} = 0.$$

Question 6. Compute the derivatives of the following functions using the derivatives of $\sin x$ and $\cos x$ along with the properties of derivatives:

a) $f(x) = \sec x$

b) $f(x) = \csc x$

c) $f(x) = \tan x$

d) $f(x) = \cot x$

Question 7. Recall that the binomial theorem gives a quick way expand high powers of monomials:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Use this and the definition of the derivative to compute the derivative of $f(x) = x^n$ for all $n \in \mathbb{N}$.

Question 8. Prove the Brouwer Fixed Point Theorem: Given any continuous function $f : [0,1] \to [0,1]$, there exists some fixed point (i.e, some point $c \in [0,1]$ such that $f(c) = c$).

Question 9. A function $f : \mathbb{R} \to \mathbb{R}$ is called K-*Lipschitz* if for every $x, y \in \mathbb{R}$, we have the inequality

$$|f(x) - f(y)| \leq K \cdot |x - y|$$

for some $K \geq 0$. In general, a function is *Lipschitz* if it is K-Lipschitz for some K.

a) Prove that a function is 0-Lipschitz if and only if f is a constant function.

b) Use the $\delta - \varepsilon$ definition to prove that any K-Lipschitz function is continuous.

c) Prove that the space of Lipschitz functions is a linear space; that is, if f and g are Lipschitz functions and $c \in \mathbb{R}$, then $c \cdot f$ is Lipschitz and $f + g$ is Lipschitz.