July 18 Workshop - Limit Calculus of Sequences and Functions

Question 1.

(a) Give an example of a bounded sequence that does not converge.

(b) Prove that every convergent sequence is bounded. You may use the fact that every finite set has a maximum and minimum element.

Question 2.

(a) Prove that the only sequences that are *both* monotone increasing and monotone decreasing are the constant sequences.

(b) Given a monotone sequence, what can you say about its boundedness. State your proposition and prove it.

Question 3. Use the $\varepsilon - N$ defintion to prove that the sequence $a_n = \frac{1}{n^2}$ converges to 0.

Question 4. Prove the following properties about sequences:

(a) If $c \in \mathbb{R}$ and we have a convergent sequence $a_n \to A$, then the new sequence $c \cdot a_n$ (where each term in a_n is multiplied by the constant c) converges; in particularly, $c \cdot a_n \to cA$.

(b) If we have two convergent sequence $a_n \to A$ and $b_n \to B$, then the new sequence $a_n + b_n$ converges as well; in particular, $a_n + b_n \to A + B$.

Question 5. Prove the following *Sequence Squeeze Theorem*:
Let a_n, b_n, and c_n be three sequences with $a_n \leq b_n \leq c_n$. If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then b_n converges as well and, in particular, $b_n \to L$.

Question 6. Let $f(x) = cx + b$. Generalize the example given in class to prove that $\lim_{x\to a} f(x) = ca + b$ using a $\delta - \varepsilon$ proof.

Question 7. Consider the circle C of radius 1. Let P_n be regular n-gon inscribed in C (that is, P_n lies completely inside of C and its vertices lie on C itself). Let $\{A_n\}$ be the sequence where the n-th term is the area of the inscribed n-gon in C. Further, let R_n be the regular n-gon in which the circle C is inscribed. Let $\{B_n\}$ be the area of this n-gon.

a) Find a closed form for each term A_n.

b) Find a closed form for each term B_n.

c) Use these closed forms to evaluate A_n and B_n for $n = 5, 10, 100$, and 1000. How much do they differ from π, the area bounded by C?

d) Argue that $\{A_n\}$ is bounded above and $\{B_n\}$ is bounded below.