August 4 Workshop - Applications of Differential Equations

Question 1. An undampened spring with a 3 kg mass is held stretched 0.6 m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds.

Question 2. A spring with mass 3 kg has damping constant 30 and spring constant 123. Find the position of the mass at time t if it starts at the equilibrium position and has an initial velocity of 2 m/s. Plot your position function.

Question 3. Assume that we have an undampened mass-spring system and we begin by pulling our mass to a displacement of x_0 and then release the mass so that at time $t = 0$, it has no initial velocity.

(a) Show that the amplitude of this system is exactly $A = x_0$ and thus the mass will go above and below its resting point by at most x_0. Why does this make sense physically?

(b) If instead of having no initial velocity, we did have some initial $x'(0) = \alpha$, what would the amplitude be?

(c) Use (b) to show that no matter whether the initial velocity is towards or away from the equilibrium, the amplitude of a spring-mass system with some non-zero initial velocity is also greater than the amplitude of a system with zero initial velocity. Is this physically believable?

Question 4. One common situation with a mass-spring system is that there is some external force that acts on a spring (for example, attaching another spring-mass system to the mass). If we have such a situation where we know the force $F(t)$ explicitly, then using Newton's Law and the discussion from lecture, we obtain the non-homogeneous, linear, second-order equation

$$mx'' + cx' + kx = F(t).$$

(a) Assume that the external force F is constant. That is, $F(t) = \alpha$. Show that $x(t) = \alpha/k$ is a particular solution for this differential equation. What does this this solution mean physically?

(b) Assume that the spring-mass system is undampened. Use (a) to find the general form for all solutions to your differential equation when $F(t) = \alpha$.

(c) Many times, the external force has a periodic behavior. Take, for example, $F(t) = F_0 \cos \omega_0 t$, where $\omega_0 \neq \omega$. Assume for this question that there is also no dampening. Using the method of undetermined coefficients, show that the general solution has the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t.$$

(d) If in (c) the external force was given by $F(t) = \cos(\omega t)$ (where now $\omega_0 = \omega$), show that the general form of the solution is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t.$$

(e) Show that for ω_0 equal to ω (as in (d)), the solutions in (d) will reinforce the natural frequency ω and thus heighten the amplitude. To show this, discuss what happens to $x(t)$ as $t \to \infty$. Such a phenomenon is known as *resonance* and is epitomized by shattering glass with a loud tone with *resonant frequency.*

Question 5. For this question, we will consider the motion of a pendulum hanging from a ceiling. This pendulum will have length L; our goal is to measure for any time t, the angle $\theta(t)$ that the pendulum makes from its resting (vertical) position. The motion of the pendulum is given by the non-linear differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

where g is the gravitational constant.

(a) Argue that for small angles θ, we may replace $\sin\theta$ by θ and turn the above differential equation into a linear second-order equation. What is this linear equation?

(b) Using this linear differential equation, find the angle of motion $\theta(t)$ of a pendulum with $L = 1$ m if θ is initially 0.2 rads and the initial angular velocity is $d\theta/dt = 1$ rad/s.

(c) What is the maximum angle from the vertical for the pendulum?

(d) What is the period of the pendulum? That is, how long does it take the pendulum to complete one full swing?

(e) At what time will the pendulum first be vertical?

(f) What is the angular velocity $d\theta/dt$ when the pendulum is vertical?

Question 6. This question investigates the role of linear algebra in solving ordinary differential equations via the *Wronskian* of two functions. Recall that in a vector space, two vectors are linearly independent if the only way to write the zero vector as a linear combination is the trivial way. When our vector space is the space of differentiable functions $C^1(\mathbb{R})$, we say that two differential functions f and g are *linearly independent* if anytime $c_1 f(t) + c_2 g(t) = 0$ for all $t \in \mathbb{R}$, then $c_1 = c_2 = 0$. Consider the (2×2)-matrix given by

$$\begin{bmatrix} f & g \\ f' & g' \end{bmatrix}.$$

The Wronskian $W(f, g)(t)$ is defined as the determinant of this matrix and is again a function of t.

(a) Prove that the functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - \cos^2 t$ are linearly dependent by finding a non-trivial linear combination of f and g that gives the zero function.

(b) Show that $f(t) = t$ and $f(t) = t^2$ are linearly independent by showing that the only way to write the zero vector as a linear combination of f and g is the trivial way.

(c) Show that if $c_1 f(t) + c_2 g(t) = 0$, then $c_1 f'(t) + c_2 g'(t) = 0$ as well.

(d) Use (c) and properties of determinants and linear independence to show that if the Wronskian $W(f, g)(t)$ is non-zero for some t_0, then f and g are linearly independent.

(e) For a second-degree linear homogeneous differential equation $ay'' + by' + cy = 0$, the three cases are when the characteristic polynomial has two distinct real roots, a repeated root, or two complex (conjugate) roots. What do solutions look like in terms of the roots of the characteristic polynomial?

(f) For each of the cases in (e), use the Wronskian to verify that the two solutions you write every solution in terms of are linearly independent.

(g) Assuming that the solution space for a second-degree homogeneous linear differential equation is 2-dimensional, why is (f) enough to ensure that your two solutions form a basis for the solution space?