August 3 Workshop - Systems of Differential Equations

Question 1. Consider the general system of first-order, linear, homogeneous differential equations with constant coefficients given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Under what conditions on a, b, c, and d will we have two distinct, real eigenvalues?

(b) Use (a) to quickly verify that the system given by

$$x'(t) = 2x + y$$

$$y'(t) = x + 2y$$

has two distinct, real eigenvalues.

(c) Find the two real, distinct eigenvalues promised in (b) and find their corresponding eigenspaces.

(d) Use (c) to draw the phase portrait of the system in (b). Is the origin a source, sink, or saddle?

Question 2. The goal of this question is to use systems of differential equations to solve second-degree ODEs. In particular, consider the second-order ODE given by

$$y'' + by' + cy = 0.$$

(a) Making the subsitition $v = y'$, write a system of differential equations for v' and y' in terms of b and c.

(b) Find the characteristic polynomial of the coefficient matrix from (a). Show that it is the same as the characteristic polynomial for the second-order differential equation.

(c) Use (a) to turn the ODE $y'' + 5y' + 4y = 0$ into a system of differential equations for v' and y'.

(d) Solve the system in (c) using systems of ODEs. What is the general solution you get for $[y(t), v(t)]$. Use the initial conditions $y(0) = 0$, $v(0) = 1$ to give a particular solution to this system.

(e) Verify that in your solution to (d), $v(t) = y'(t)$.

(f) Solve $y'' + 5y' + 4y = 0$ using the second-order methods of yesterday. Use the intial conditions $y(0) = 0$ and $y'(0) = 1$ to give a particular solution. Does your solution match the one from (d)?

Question 3. Consider the system of differential equations given by

$$x'(t) = 10x - 18y$$

$$y'(t) = 6x - 11y$$

(a) Find the eigenvalues and corresponding eigenspaces for the coefficient matrix A.

(b) Use (a) to find a general solution for the above system of ODEs.

(c) Furthemore, use (a) to sketch the phase portrait of this system.

(d) If we wanted to find the particular solution that goes through the point $(-1, 3)$, what initial values would we set-up?

(e) Use the initial values from (d) to find the explicit form of the particular solution. Plot your answer on your previously drawn phase portrait.

(f) Classify the origin as a sink, saddle, or source equilibrium. What can you say about the dynamics of the particular solution you found in (e)? Does it tend towards the origin or away from the origin?

(g) Explain how your particular solution would behave if in (d) we had asked you to find a solution that goes through a point on one of the eigenspaces.

Question 4. Consider the vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(a) Find a system of differential equations that has its eigenspaces spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, respectively. Sketch the phase portrait of this system. Is the origin a sink, saddle, or source?

(b) Find a system of differential equations that has its eigenspaces spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$ with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$, respectively. Sketch the phase portrait of this system. Is the origin a sink, saddle, or source?

(c) Find a system of differential equations that has its eigenspaces spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$ with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$, respecitvely. Sketch the phase portrait of this system. Is the origin a sink, saddle, or source?

(d) Find a system of differential equations that has its eigenspaces spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$ with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 0$. Sketch the phase portrait of this system.

Question 5. In this example, we will learn a technique called *the method of undtermined coefficients* to solve a non-homogeneous system of non-homogeneous ODEs. Consider the following system of equations:

$$x'(t) = x + 2y + 2t$$
$$y'(t) = 3x + 2y - 4t.$$

To simplify notation, we will set the following notation for our solution vector $\mathbf{s}(t) = [x(t), y(t)]$.

(a) Re-write the above system of ODEs in terms of the following matrix equation

$$\mathbf{s}'(t) = A\mathbf{s}(t) + t\mathbf{g},$$

where A is a (2×2)-coefficient matrix, $\mathbf{g}$ is a 2-vector, and $\mathbf{s}'(t) = [x'(t), y'(t)]$.

(b) Assume that we have found a solution $\mathbf{s_h}(t)$ for the corresponding homogeneous equation

$$\mathbf{s}'_\mathbf{h}(t) = A\mathbf{s_h}(t)$$

and a particular solution $\mathbf{s_p}(t)$ for the original non-homogeneous equation

$$\mathbf{s}'_\mathbf{p}(t) = A\mathbf{s_p}(t) + t\mathbf{g}.$$

Show that their sum $\mathbf{s_h}(t) + \mathbf{s_p}(t)$ is also a solution to the non-homogeneous equation. Thus, to find any solution to the non-homogeneous equation, we must find a particular solution $\mathbf{s_p}$ and the homogeneous solutions $\mathbf{s_h}$.

(c) Find the general solution to the homogeneous equation $\mathbf{s}'_\mathbf{h}(t) = A\mathbf{s_h}(t)$ using the methods learned in lecture.

(d) The method of undetermined coefficients necessitates guessing a form for the particular solution $\mathbf{s_p}$. Since the non-homogeneous part is linear, let us consider a solution of the form

$$\mathbf{s_p}(t) = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{a}t + \mathbf{b}$$

Assuming that $\mathbf{s_p}(t)$ is of this form, find the coefficients a_1, a_2, b_1 and b_2 that make the differential equation hold.

(e) Using your homogeneous solution $\mathbf{s_h}(t)$ from (c) and your particular solution $\mathbf{s_p}(t)$ from (d), find the general form for all solutions to our original system $\mathbf{s}(t) = A\mathbf{s} + t\mathbf{g}$.