August 2 Workshop - Exact and Second-order Differential Equations

Question 1. Solve the initial value problem given by

$$\cos x - x\sin x + y^2 + 2xyy' = 0$$

and $y(\pi) = 1$.

Question 2. It is a well-known fact from multivariable calculus that mixed partial fractions are equal; that is, given some C^2 function $f(x, y)$ in two variables x and y then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(a) Consider a differential equation given by $M(x, y) + N(x, y)y' = 0$. Use the above fact to show that if

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

then our differential equation is not exact.

(b) Show that exactness is a fragile condition by showing that

$$\cos y + (y^2 - x\sin y)y' = 0$$

is exact (by solving it through exact methods) while

$$\cos y + (y^2 + x\sin y)y' = 0$$

is not exact (and thus not solvable through exact methods).

Question 3.

a) Give the general form for the solution of the differential equation

$$y'' + y' - 2y = 0.$$

b) Give the general form for the solution of the differential equation

$$y'' + 2y' + 2y = 0.$$

Question 4. Solve the initial value problem

$$y'' - 4y' + 4y = 0$$

with $y(0) = 1$, $y'(0) = 1$.

Question 5. Consider the *inhomogenous equation*

$$ay'' + by' + cy = g(t).$$

To every such equation there corresponds a *homogeneous* equation where we take $g(t) = 0$:

$$ay'' + by' + cy = 0.$$

Recall further that the space of solutions to a homogeneous linear ODE is a vector space.

(a) Let y_h be a solution to the homogeneous equation and y_p a solution to the inhomogeneous equation. Show that $y_h + y_p$ is a solution to the inhomogeneous equation.

(b) Let y_1 and y_2 be any solutions to the inhomogeneous differential equation. Show that $y_1 - y_2$ is a solution to the homogeneous linear eqution.

(c) Use (b) to argue that any two solutions to the inhomogeneous linear equation differ by a solution to the homogeneous equation. Thus, we may obtain all solutions to the inhomogeneous differential equation by finding a particular solution y_p to the inhomogeneous equation and adding a solution to the homogeneous equation.

Question 6. This exercise is meant to give a step-by-step approach to solving an inhomogenous second order differential equation by using the *Method of Undetermined Coefficients.* Consider the differential equation

$$y'' - 5y' + 6y = t.$$

(a) Find the general solution for the homogenous equation

$$y'' - 5y' + 6y = 0.$$

(b) Assume that a solution to our inhomogenous differential equation has the form $y(t) = At + B$. Find the coefficients A and B.

(c) Recall that we proved in the above workshop question that any solution to an inhomogeneous linear ODE is given by adding particular solution to the inhomogeneous ODE and a solution to the homogeneous ODE. Use (a) and (b) to give the general form of the solution to the inhomogeneous equation.

Question 7. In this question, we will give a differential definition of hyperbolic sine and cosine, two functions crucial to understanding curved geometries.

(a) If we were to search for a function whose second derivative is equal to itself, what differential equation should we set up?

(b) Solve the differential equation in (a).

(c) Hyperbolic sine (written $\sinh t$) is the solution to the above ODE that has the further property that $\sinh(0) = 0$ and $\sinh'(0) = 1$. Use these initial values to find $\sinh t$ explicitly.

(d) Hyperbolic cosine (written $\cosh t$) is the solution to the above ODE that has the further property that $\cosh(0) = 1$ and $\cosh'(0) = 0$. Use these initial values to find $\cosh t$ explicitly.

(e) Use the explicit forms you obtained in (c) and (d) to show that $\sinh t$ is the derivative of $\cosh t$ and vice versa.

(f) Show that $\sinh t$ and $\cosh t$ satisfy the below relations:

$$\cosh t + \sinh t = e^t \quad \text{and} \quad \cosh t - \sinh t = e^{-t}.$$

(g) Show that $\sinh t$ and $\cosh t$ satisfy the below trigonometric-like relations:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$$