August 1 Workshop - Ordinary Differential Equations

Question 1.

(a) Give the family of solutions for the differential equation

$$y' - 2y = 4 - t.$$

(b) Find the particular solution corresponding to the initial value $y(0) = 2$.

Question 2. Give a general solution for the differential equation

$$y' = e^{5x - \ln(y^2)}.$$

Question 3. Recall that a linear ordinary differential equation of degree n is given by

$$a_n(t) \cdot y^{(n)} + a_{n-1}(t) \cdot y^{(n-1)} + \cdots + a_0(t) \cdot y = g(t),$$

where $a_k(t)$ are differentiable functions. Furthermore, a linear ODE is called *homogeneous* if $g(t) = 0$.

(a) Prove that the constant zero function $y(t) = 0$ is always a solution to a homogeneous linear ODE.

(b) Prove that the set of solutions to a linear, homogeneous ODE is a vector space by showing that if $y_1(t)$ and $y_2(t)$ are solutions and $\alpha \in \mathbb{R}$, then $y_1(t) + y_2(t)$ and $\alpha y_1(t)$ are also solutions.

(c) Show that when the degree n of our linear, homogeneous ODE is 0, then the solution space is zero-dimensional (and thus contains only a single element). What is this single element?

(d) Show that a first-degree linear, homogeneous ODE is separable and solve it for a general $a_1(t)$ and $a_0(t)$.

Question 4. Electrical circuits are often described in terms of differential equations. Kerchoff's Law gives a way to describe the resistance R, current I, inductance L, and voltage V of a circuit by the differential equation

$$L\frac{dI}{dt} + RI = V(t).$$

(a) Suppose that for a circuit, the resistance is 12Ω (Ω stands for *Ohms*) and the inductance is $4H$ (H stands for *Henrys*). If a battery gives a constant voltage of 60 V and a switch is closed so the current starts with $I(0) = 0$, find $I(t)$, the current after one second $I(1)$.

(b) In the circuit described in (a), what will happen to the current as $t \to \infty$?

(c) If instead of a battery, we were to use a generator that produced $V(t) = 60 \sin 30t$ volts (and the same R and L as above), what is the current $I(t)$?

(d) What happens to the circuit from (c) if $t \to \infty$?

Question 5. Falling objects may also be desribed in terms of first order differential equations. An object with mass m is dropped from rest. If $s(t)$ described the distance dropped after t seconds, then its speed is given by $v(t) = s'(t)$ and its acceleration by $a(t) = v'(t) = s''(t)$. Assuming that air resistance is proportional to the velocity and that g is the acceleration due to gravity, then the downward force on the object is given by $mg - cv$, where c is a constant determined by air resistance.

(a) Newton's Laws of Motion stipulate that force is equal to the product of mass and acceleration. What is the force of downward falling object in terms of its velocity?

(b) Newton's Laws also indicate that the force of the object (found in (a)) must be equal to the downward force described in the prompt. What is the differential equation that Newton's Laws give?

(c) Solve the above differential equation.

(d) What is the limiting velocity? This velocity is known as *terminal velocity* and greatly depends on the air resistance constant c.

(e) Find the distance that the object has fallen after t seconds.

Question 6. A *Bernoulli differential equation* (named after James Bernoulli) is of the form

$$\frac{dy}{dt} + P(t)y = Q(t)y^n.$$

In this question, we will develop a technique called *substitution* to turn these *non-linear* differential equations into *linear* differential equations.

(a) Show that when $n = 0, 1$ that the Bernoulli equation is already linear.

(b) If $u = y^{1-n}$, show that

$$\frac{du}{dt} = (1-n)y^{-n}\frac{dy}{dt}.$$

(c) Use (b) to show that

$$\frac{dy}{dt} = \frac{u^{n/(1-n)}}{1-n}\frac{du}{dt}.$$

(d) Use (c) to turn the Bernoulli equation into the differential equation

$$\frac{u^{n/(1-n)}}{1-n}\frac{du}{dt} + P(t)u^{1/(1-n)} = Q(t)u^{n/(1-n)}.$$

(e) Use (d) to turn the Bernoulli equation into the linear differential equation

$$\frac{du}{dt} + (1-n)P(t)u = Q(t)(1-n).$$

(f) Use (e) to solve the nonlinear differential equation

$$y' - \frac{2y}{t} = -t^2y^2.$$