Chapter 8

Thursday, July 21, 2011

8.1 Applications of Calculus: Fourier Analysis

8.1.1 The Idea of Fourier Analysis

In the homework problems, we have seen via Taylor polynomials that functions can many times be approximated by a sequence of other functions of higher and higher complexity. With Taylor polynomials, we use high-order derivatives of $f(x)$ at a given point a to construct polynomials $T_n(x)$ of increasing degree that frequently begin to mimic the overall behavior of f. Though we have not formally defined the convergence of a sequence of functions, we can intuitively see that for many functions, as the degree of our Taylor polynomial grows, the the gap between $T_n(x)$ and $f(x)$ shrinks.

We continue in this vein by constructing a sequence of functions that seem to converge to certain functions f. In contrast to using the derivatives of $f(x)$, the Fourier expansion of f will use integration to find our approximating functions. Furthermore, we must restrict ourselves to functions that are *periodic*, or repetitive.

8.1.2 Periodic Functions

Fourier analysis utilizes the two trigonometric functions $\sin x$ and $\cos x$ to build complicated functions frequently observed in acoustics, optics, and mechanics. A crucial property of both $\sin x$ and $\cos x$ is that their graphs repeat every 2π. In general, we say that a function is ***periodic*** if there exits some $k > 0$ such that $f(x + k) = f(x)$ for all $x \in \mathbb{R}$. These are precisely the graphs that look exactly the same if you look at x or at $x + k$ and thus have a repeating pattern. If f is periodic, then the smallest $k > 0$ for which $f(x + k) = f(x)$ holds is called the ***period*** of f. Thus, both $\sin x$ and $\cos x$ are periodic of period 2π.

Because of their repetition, to know what happens to $f(x)$ for all values of x, we only need to understand f on some bounded interval $[a, a + k]$ (for any a). In fact, it is usually easiest to understand f on the domain $[-k/2, k/2]$; the rest of f can be understood by repeating what happened in $[-k/2, k/2]$.

8.1.3 The Fourier Expansion

The goal of the Fourier expansion is to approximate f using $\sin(nx)$ and $\cos(mx)$ for increasing values of n and m. In fact, we wish to use functions of the form

$$\hat{f}_n = \frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos(kx) + b_k \sin(kx)\right].$$

The coefficients a_i and b_i are known as the *harmonics* of f and have physical and acoustic interpretations.

Essentially, a harmonic expansion $\hat{f}_n$ of f is a decomposition of f into simpler periodic pieces. We will only define the harmonics for functions of period 2π; for functions of arbitrary period, a similar formulation is available (but not mentioned in these notes). To this end, we define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Of course, the idea is that as we add more harmonics a_n and b_n into approximation, $\hat{f}_n$ will approach f. Let's demonstrate the power of the Fourier expansion with an example. Consider the *square wave* function given on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & -\pi \le x < 0 \end{cases}$$

By periodicity, we can translate this every 2π to obtain graph of $f(x)$ for all $\mathbb{R}$.

To find the harmonics of the function, we must integrate f against $\sin(nx)$ and $\cos(nx)$. We see that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(0x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

In general, we find that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\,dx = 0.$$

In contrast, we have that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx =$$

$$\left[\frac{1}{\pi n}\cos(nx)\right]_{-\pi}^{0} + \left[-\frac{1}{\pi n}\cos(nx)\right]_{0}^{\pi} =$$

$$\left(\frac{1}{\pi n} - \frac{(-1)^n}{\pi n}\right) + \left(-\frac{(-1)^n}{\pi n} + \frac{1}{\pi n}\right) = \frac{2}{n\pi} - \frac{2(-1)^n}{n\pi} = \frac{2(1-(-1)^n)}{n\pi}.$$

Notice that the last expression gives us 0 when n is even and $\frac{4}{n\pi}$ when n is odd. Thus, we have the following Fourier expansions of $f(x)$:

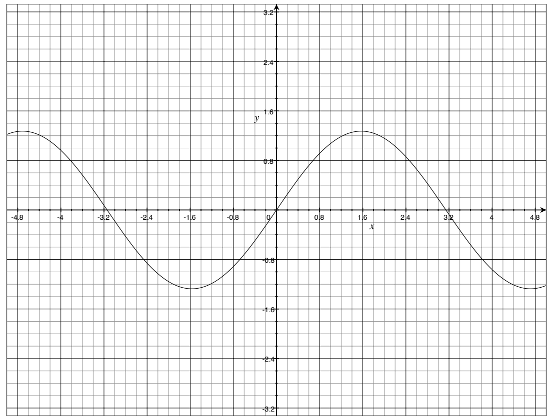
$$\hat{f}_1(x) = \frac{4}{\pi}\sin x;$$

$$\hat{f}_3(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right);$$

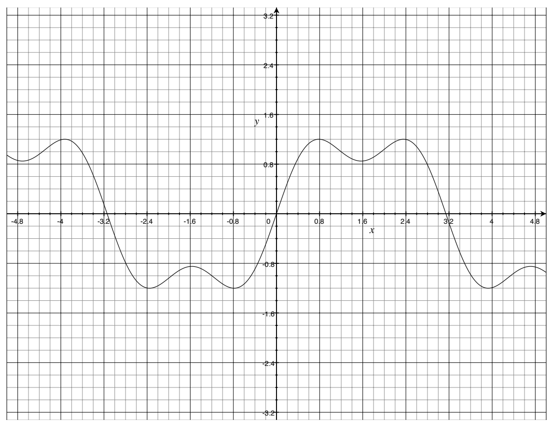
$$\hat{f}_5(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right);$$

$$\hat{f}_7(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x\right).$$

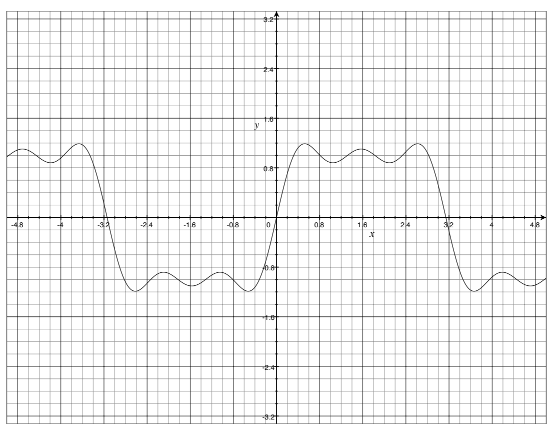
Below are supplied the plots of these functions. Notice that as n increases, the graphs of $\hat{f}_n$ tend toward the graph of $f(x)$.



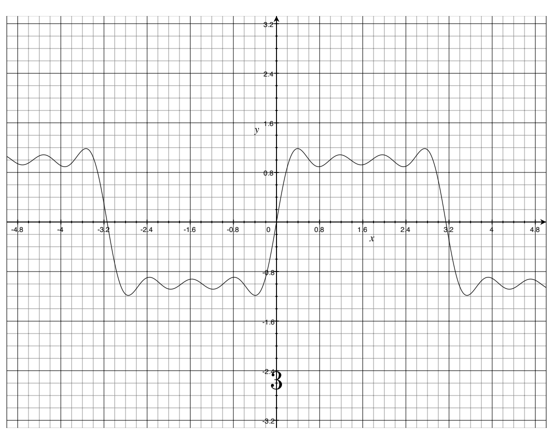
(a) $\hat{f}_1(x)$



(b) $\hat{f}_3(x)$



(c) $\hat{f}_5(x)$



(d) $\hat{f}_7(x)$