Chapter 7

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7.1 Applications of the Derivative

7.1.1 The Mean Value Theorem

The Mean Value Theorem is perhaps the most important theorem involving differentiation. To motivate the theorem, let us imagine that we are driving a car from time a to time b in a differentiable way (e.g., no sudden braking or accelerating). Let $f(t)$ be our *position function*, which tells us where our car is at time t. If someone asks you "how fast did you travel from time a to time b," there are two ways to answer: in terms of averages or instantaneously. If they only care about your average speed, then you would simply take the total distance travelled $(f(b) - f(a))$ and divide that by the total time $(b - a)$. The ratio

$$\frac{f(b) - f(a)}{b - a}$$

is known as the *mean* or *average* velocity. However, if the person wants to know your instantaneous velocity, then you would simply supply the derivative $f'(t)$ of f. This would give at any time t the exact velocity as read on your speedometer as time t. Of course, as you drove from time a to time b, your velocity varied (depending on traffic, stoplights, etc.). These two different answers to the same question beg a natural questions: is there always a time t between a and b such that our instantaneous velocity is exactly the same as the average velocity? The ***Mean Value Theorem*** guarantees that the answer is yes.

Theorem - The Mean Value Theorem. Let f be a function differentiable on $[a, b]$. There exists some c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In graphical terms, this says that for a differentiable function f, there exists at least one c such that the tangent line at c has the same slope as the secant line between $(a, f(a))$ and $(b, f(b))$. However, just like with the Intermediate Value Theorem, this theorem is *non-constructive*, which means that it does not tell you which c it is guaranteeing (only that it exists). The mere existence, though, is enough to produce some very important results.

7.1.2 Ramifications of the Mean Value Theorem

One of the most impressive corollaries of the Mean Value Theorem is that it gives us a graphical interpretation of the derivative. In particular, it tells us that the sign of $f'(x)$ tells us if f is increasing or decreasing. First, let us define exactly what we mean by these terms. A function is said to be ***increasing*** if whenever $x < y$, $f(x) \leq f(y)$. Similarly, a function is said to be ***decreasing*** if whenever $x < y$, $f(x) \geq f(y)$. These definitions indicate that increasing functions are going up, while decreasing functions are going down.

Proposition. Let $f(x)$ be a differentiable function of $[a, b]$.

(a) If $f'(x) = 0$ for all $x \in [a, b]$, then $f(x)$ is a constant function.

(b) If $f'(x) \leq 0$ for all $x \in [a, b]$, then $f(x)$ is a decreasing function.

(c) If $f'(x) \geq 0$ for all $x \in [a, b]$, then $f(x)$ is an increasing function.

Proof. To prove (a), we must show that $f(x) = f(y)$ for all $x, y \in [a, b]$. By the Mean Value Theorem and the fact that $f'(c) = 0$ for all $c \in [a, b]$, we have that

$$0 = f'(c) = \frac{f(y) - f(x)}{x - y}.$$

Assuming $x \neq y$, then we have that $f(y) - f(x) = 0$ and thus $f(x) = f(y)$. Thus, f is a constant function.

To prove (b), we must show that if $x < y$ and $f'(c) \leq 0$ for all $c \in [a, b]$, then $f(x) \geq f(y)$. By the Mean Value Theorem, we have that

$$0 \geq f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $x < y$, then $y - x > 0$ and so $0 \geq f(y) - f(x)$. Thus, $f(x) \geq f(y)$ and f is decreasing.

To prove (c), we mimic the proof of (b). By the Mean Value Theorem, we have that

$$0 \leq f'(c) = \frac{f(y) - f(x)}{y - x}.$$

As above, $f(y) \geq f(x)$ and thus f is increasing.

7.2 The Integral

7.2.1 The Definite Integral as Area

Consider a function $f(x)$ which is positive on the interval $[a, b]$. Then, $f(x)$ lies above the x-axis and we can talk about the region bounded by the x-axis on the bottom, the graph of $f(x)$ on top, $x = a$ on the left and $x = b$ on the right. We want to produce an easy way of computing the area of this region. Since $f(x)$ is curved, our best approach to computing this area is to approximate the area by thin rectangles stacked next to each other.

First, we choose a partition of the interval $[a, b]$ into n points. To do this, we consider n points equally spaced apart; as they are equally spaced, they will have to be $\Delta x = \frac{b-a}{n}$ apart. Thus, if we start at a, the next equally spaced point will be at $a + \Delta x = a + \frac{b-a}{n}$. Going over one more Δx, we move to

$a + 2\frac{b-a}{n}$. Applying this process n times, we see that the 0-th point x_0 is at a and, in general, the i-th point x_i to be given by

$$x_i = a + i\Delta x = a + i\frac{b-a}{n}.$$

Wee see that the n-th point is given by

$$x_n = a + n\Delta x = a + n\frac{b-a}{n} = a + b - a = b.$$

We will now use these points to build a rectangle based at this point with height given on the curve $f(x)$. In particular, consider the first rectangle whose bottom left corner is at x_0 and bottom right corner is at x_1 and has height $f(x_0)$. Thus, the area of this rectangle is base times height. The base length is given by

$$x_1 - x_0 = (a - \Delta x) - a = \Delta x,$$

and the height is given by $f(x_0)$. So, the area of this rectangle is given by

$$f(x_0) \cdot \Delta x.$$

If we run this construction for any $i \leq n - 1$ to build the i-th rectangle with base between x_i and x_{i+1} and height $f(x_i)$, we can again calculate its area as being

$$f(x_i) \cdot \Delta x.$$

These rectangles now fit into the graph of $f(x)$ as approximating the area under the curve. We note that we must stop at the $n-1$ rectangle since this the n-th rectangle would extend to x_{n+1}, which is outside of $[a, b]$.

Now that we have these rectangles, let's find a closed form (depending on n) of what this approximation will be. Since the union of all rectangles will have area close to the area bound by the curve. Thus, we add up all the areas of the rectangles to obtain

$$I_n(f) = \sum_{i=0}^{n-1} f\left(a + i\Delta x\right)\Delta x = \sum_{i=0}^{n-1} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}.$$

Since Δx is constant (with respect to i), we can pull it out of the sum to obtain the *Left Approximation*:

$$I_n(f) = \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a + i\frac{b-a}{n}\right).$$

In the above construction, only the upper left corner of each rectangle touched the graph. If we instead choose to have the upper right corner of each rectangle touch the graph, we must alter our construction by starting at $x_1 = a + \Delta x$ (instead of at a. Also, we must end at $x_n = a + nb - an = a$ (instead of at $x_{n-1} = a + (n-1)\frac{b-a}{n} = b - \frac{b-a}{n}$). Thus, the area of the rectangles will still be $f(x_i) \cdot \Delta x$, but our i will range from 1 to n. Thus, this *Right Approximation* gives us

$$I^n(f) = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right).$$

Whichever approximation we wish to use, it is clear that our choice of how many partitions of $[a, b]$ we take (given by n) will affect our approximation of the area under the curve. The more partitions that we have, the less error our approximation will have. These approximations $I_n(f)$ and $I^n(f)$ are both sequences of real numbers. If any one of these limits converges to a real number, then this limit is the area bound by the curve and our function $f(x)$ is called *integrable*. This quantity is called the *definite integral* and is denoted by

$$\lim_{n\to\infty} I_n(f) = \int_a^b f(x)\,dx = \lim_{n\to\infty} I^n(f).$$

Of course, it is a good exercise to show that $\lim_{n\to\infty} I_n(f)$ converges if and only if $\lim_{n\to\infty} I^n(f)$.

Notice that the above construction and calculation does not require that $f(x)$ lie above the x-axis on $[a, b]$. Thus, the definite integral generalizes a concept of *signed area*, where portions of the graph lying below the x-axis bound *negative area*.

7.2.2 The Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus is both surprising and practical. It is surprising because it links intimately the separate concepts of the derivative and integral. It is practical because it gives an explicit way of computing the area under a curve.

The key concept here is that of an *antiderivative*. Given a function $f(x)$, if there exists a differentiable function $F(x)$ such that $F'(x) = f(x)$, then $F(x)$ is called an antiderivative of $f(x)$. Notice that if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$ (where is c is a constant) is also an antiderivative. Thus, antiderivatives of $f(x)$ are not unique; however, it is true that any two antiderivatives differ by a constant. Below is the statement of the Fundamental Theorem of Calculus.

Theorem. [The Fundamental Theorem of Calculus] Let $f(x)$ be an integrable function and $F(x)$ any antiderivative of $f(x)$ (i.e, $F'(x) = f(x)$). Then,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Thus, we can compute the area bounded by the curve of $f(x)$ by finding an antiderivative $F(x)$ and plugging in the endpoints a and b. Of course, if we choose a different antiderivative $G(x) = F(x) + c$, then the computation of the definite integral with either $F(x)$ or $G(x)$ will yield the same answer:

$$\int_a^b f(x)\,dx = G(b) - G(a) = (F(b) - c) - (F(a) - c) = F(b) - F(a).$$

7.2.3 Computing Antiderivatives.

The best way to compute antiderivatives is to take guesses of what they may be and verify they are indeed antiderivatives by taking a derivative.

Monomials. Consider the function $f(x) = x^n$ for $n \neq -1$. Consider the potential antiderivatives $F(x) = \frac{x^{n+1}}{n+1}$. Taking a derivative, we see that

$$F'(x) = \frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{n+1}{n+1}x^n = x^n = f(x).$$

Thus, all possible antiderivatives of $f(x) = x^n$ is given by

$$F(x) = \frac{x^{n+1}}{n+1} + c.$$

Notice that since $n \neq -1$, we are never dividing by zero.

Trigonometric Functions. We know that $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$. Thus, we can see directly that the antiderivatives of $\cos x$ is given by

$$F(x) = \sin x + c,$$

and the antiderivatives of $\sin x$ are given by

$$F(x) = -\cos x.$$

Exponential. Since the derivative of e^x is itself, we see that the antiderivative of e^x is given by

$$F(x) = e^x + c.$$

7.2.4 Properties of the Integral.

Many of the useful properties of integrals come from using the Fundamental Theorem of Calculus in conjunction with the properties of derivatives.

Properties of the Integral. Assume that f and g are differentiable and integrable functions and that c is any real constant.

a) [Integrals of Scaled Functions] We can pull out scalars from integrals:

$$\int_a^b c \cdot f(x)\, dx = c \int_a^b f(x)\, dx.$$

b) [Additivity of the Integral] The integral of a sum is the sum of the integrals:

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b f(x)\, dx.$$

c) [Integration by Parts]

$$\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du.$$

d) [u-substitutions]

$$\int_a^b f(u(t)) \cdot u'(t)\, dt = \int_{u(a)}^{u(b)} f(x)\, dx.$$