Chapter 6

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6.1 Continuity

6.1.1 Defining Continuity of a Function.

Yesterday, we saw that the limit $\lim_{x\to a} f(x)$ of particularly nice functions was essentially obtained by plugging in $x = a$. The types of functions where we can do this are called *continuous*.

Let $f(x)$ be a function. We say that the function $f(x)$ is ***continuous at*** a if the following two conditions hold:

1) $f(a)$ exists (i.e., $f(x)$ is defined at a)

2) $\lim_{x\to a} f(x) = f(a)$.

We say that $f(x)$ is a ***continuous function*** if it is continuous at every point $a \in \mathbb{R}$.

Essentially, continuous functions are precisely those where we can compute their limits in the simplest way possible: just plug in the value a into the function $f(x)$. This may also be interpreted in terms of the geometry of the graph of $f(x)$. The first condition asserts that $f(x)$ has no holes in its graph. The second condition asserts that $f(x)$ has no jumps in its graph.

6.1.2 Properties of Limits and Continuity

Knowing how limits behave under the usual operations of addition, subtraction, multiplication, and division help us to generate more examples of continuous functions.

Properties of Limits. Assume that $f(x)$ and $g(x)$ are two functions with $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

a) If $c \in \mathbb{R}$, then $\lim_{x\to a} (c \cdot f(x)) = c \cdot \lim_{x\to a} f(x) = c \cdot L$.

b) $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.

c) $\lim_{x\to a} (f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$.

d) If $g(x)$ and $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Since the definition of continuity is based explicitly on the limit, the above properties of limits give us similar properties for continuous functions.

Properties of Continuous Functions. If $f(x)$ and $g(x)$ are continuous at a point a, then the following hold:

a) For any $c \in \mathbb{R}$, the function $c \cdot f(x)$ is also continuous at a.

b) The sum function $f(x) + g(x)$ is also continuous at a.

c) The product function $f(x) \cdot g(x)$ is continuous at a.

d) If $g(a) \neq 0$, then the quotient function $\frac{f(x)}{g(x)}$ is continuous at a.

6.1.3 Applications of Continuity

The idea of a continuous function having no holes in its graph can be used to obtain several statements about continuous functions that seem obvious.

Intermediate Value Theorem. Let $f(x)$ be a continuous function $f : [a, b] \to \mathbb{R}$. Then for every y between $f(a)$ and $f(b)$, there exists some $c \in [a, b]$ such that $f(c) = y$.

Thus, a continuous function cannot skip points. A straightforward corollary of this is the following:

Bolzano's Theorem. If $f(x)$ is a continuous function which changes signs, then $f(x)$ has a root.

In fact, Bolzano's theorem gives us an *analytic* proof of the fact that every odd-degree polynomial has at least one real root. Since our polynomial $p(x)$ is odd-degree, we can always find points $x_1, x_2 \in \mathbb{R}$ where $p(x_1) > 0$ and $p(x_2) < 0$ by looking at points postive and negative enough. We have already proved this statement *algebraically* using the fact that roots of real polynomials come in complex conjugate pairs; our analytic proof gives the more graphical interpretation that (continuous) polynomials must cross the x-axis at some point.

6.2 The Derivative.

6.2.1 The Idea of a Derivative

Now that we have the notion of a limit, we will be focusing on a very special kind of limit, the *derivative*. First, we give the definition of a derivative and then see how this relates to limits of slopes and instantaneous rates of change.

Given a function $f(x) : \mathbb{R} \to \mathbb{R}$, we say that $f(x)$ is differentiable at a if the limit $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists. This limit, if it exists, is the *derivative of* $f(x)$ *at* a and is denoted by $f'(a)$. Thus,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Now, we will try to understand what this definition of a derivative has to do with limits of slopes. Intuitively, the derivative of $f(x)$ at a point a (also known as $f'(a)$) should be the instantaneous rate of change of f at a. In other words, if we had a line *tangent* to $f(x)$ at a, then the slope of this line is understood to be its derivative. How do we see this from the previous definition of the derivative that we gave. We can approximate this tangent line by a sequence of *secant* lines. To find these secant line, we consider the line through the point (a, f(a)) and $(a+h, f(a+h))$, where h is small. Notice that both of these points are on the curve of $f(x)$. This slope of this line is given by "rise over run":

$$\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}.$$

As our $h \to 0$, both the top and bottom terms tend to zero; however, it is precisely this ratio of how fast rise and run go to zero that will give us the slope of the tangent line.

Thus, at each point $a \in \mathbb{R}$ where this derivative limit exists, finding this derivative $f'(a)$ gives a function called *the derivative function* $f'(x)$. If this $f'(x)$ exists at all points, then our function f is called *differentiable.* It's called the derivative of $f(x)$ because it is derived only from f using this limit. This derived function $f'(x)$ lets us know what the slope of the tangent line is at $(x, f(x))$.

The notation $f'(x)$ for the derivative of a function is known as *Newton's notation for a derivative.* Historically, the derivative was simultaneously and independently defined by Liebniz. His notation for the derivative of a function $f(x)$ is given by $\frac{df}{dx}$. This notation reminds us that the derivative was taken by finding the slope of secant lines of the graph $\frac{\Delta f}{\Delta x}$ and taking $\Delta x \to 0$. We will interchange between these two common notations.

6.2.2 Computing a Derivative

In practice, finding this limit may be difficult. Here is a good starting example.

Example. Compute the derivative of $f(x) = b$, the constant function.

Computation. Using the limit definition of a derivative, we will compute $f'(a)$. Then,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Since $f(x) = b$, we obtain

$$\lim_{h\to 0} \frac{b-b}{h} = \lim_{h\to 0} 0.$$

Thus, we are taking the limit of the constant 0 function as $h \to 0$. Thus, the limit will be 0 and we have that $f'(a) = 0$. Since this is true for any a we choose, our derivative function will be $f'(x) = 0$. Intuitively, this is clear because the constant graph is horizontally flat and thus any tangent line will also be horizontally flat and have slope 0.

Here is the computation of the derivative of any linear function $f(x) = mx + b$.

Example. Compute the derivative of $f(x) = mx + b$.

Computation. Again, we work with the definition as a limit. Using our function $f(x)$, we have that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(m(a+h)+b)-(ma+b)}{h} =$$

$$\lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m.$$

Of course, this makes geometric sense because the tangent lines to the graph of the linear function $f(x) = mx + b$ will have tangent lines the same slope as the function itself, namely m.

Further, we can see that the previous example is a special case of this example. The constant function $f(x) = b$ is the linear function with $m = 0$ since $f(x) = 0x + b = b$. Of course, then $f'(x) = m = 0$.

6.2.3 Some Common Derivatives

We will find the derivatives for a variety of basic functions. Using properties of the derivative, we can then compute derivatives for more complicated functions.

Example - Monomials. For any $n \in \mathbb{R}$, the derivative of the monomial x^n is given by multiplying by n and reducing the exponent to $n - 1$. In other words,

$$\frac{d}{dx} x^n = n \cdot x^{n-1}.$$

Example - The Trigonometric Functions. The functions $\sin x$ and $\cos x$ are related by their derivatives. Specifically, the derivative of $\sin x$ is $\cos x$:

$$\frac{d}{dx} \sin x = \cos x.$$

Further, the derivative of $\cos x$ is $-\sin x$:

$$\frac{d}{dx} \cos x = -\sin x.$$

Example - The Exponential Function. The exponential function $f(x) = e^x$ is the unique function that with that property that it is its own derivative:

$$\frac{d}{dx} e^x = e^x.$$

In fact, we can generalize this for any base number b. If $g(x) = b^x$ where $b > 0$, then

$$\frac{d}{dx} b^x = (\ln b) \cdot b^x.$$

Indeed, when $b = e$, this reduces to our above derivative of e^x.

Example - The logarithm. The natural logarithm (or logarithm with base e) has the following derivative:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

6.2.4 Properties of Derivatives

Let $f(x)$ and $g(x)$ be two differentiable functions with derivatives $f'(x)$ and $g'(x)$, respectively.

a) [Derivatives of Scaled Functions] The derivative of $c \cdot f(x)$ is $c \cdot f'(x)$:

$$(c \cdot f(x))' = c \cdot f'(x).$$

b) [Additivity of the Derivative] The derivative of the sum function $f(x) + g(x)$ is $f'(x) + g'(x)$:

$$(f(x) + g(x))' = f'(x) + g'(x).$$

c) [The Product Rule] The derivative if the product function $f(x) \cdot g(x)$ is $f'(x) \cdot g(x) + f(x) \cdot g'(x)$:

$$((f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

d) [The Quotient Rule] If $g(x)$ is never zero, then the derivative of $\frac{f(x)}{g(x)}$ is given as follows:

$$\frac{f(x)}{g(x)}' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}.$$

e) [The Chain Rule] If we have the composition of functions $f(g(x))$, then its derivative is given as follows:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

6.2.5 Computing some derivatives

We will use the basic derivatives that we know and the properties above to compute derivatives of more complicated functions.

Examples.

1) Let $f(x) = 5x^3 - 7x + 2$. Then,

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(5x^3 - 7x + 2\right).$$

We may use the additivity and scaling of derivatives to compute the $f'(x)$:

$$\frac{d}{dx} 5x^3 - 7x + 2 = \frac{d}{dx} 5x^4 - \frac{d}{dx} 7x + \frac{d}{dx} 2 =$$

$$5 \frac{d}{dx} x^4 - 7 \frac{d}{dx} x + \frac{d}{dx} 2 = 5 \cdot 4x^3 - 7 \cdot 1 + 0 = 20x^3 - 7.$$

2) Let $h(x) = x^2 \sin x$. Then, we use the Product Rule (with $f(x) = x^2$ and $g(x) = \sin x$ so that $h(x) = f(x) \cdot g(x)$). thus,

$$\frac{d}{dx} = \frac{d}{dx}\left(x^2\right) \cdot \sin x + x^2 \cdot \frac{d}{dx} \sin x =$$

$$2x \sin x + x^2 \cos x.$$

3) Consider $h(x) = \cos(x^3)$. Then, we see this as the composition of two functions: $h(x) = f(g(x))$ where $f(x) = \cos x$ and $g(x) = x^3$. The chain rule tells us that we need to evaluate $f'(x) = -\sin x$ and $g'(x) = 3x^2$ and plug this into the function $h'(x) = f'(g(x)) \cdot g'(x)$. Since $f'(x) = -\sin x$, then $f'(g(x)) = -\sin(x^3)$. Thus, the derivative is given by

$$\frac{d}{dx} \cos(x^3) = -\sin(x^3) \cdot (3x^2) = -3x^2 \sin(x^3).$$

4) Consider the quotient function

$$h(x) = \frac{e^x}{\sin x}.$$

We use the Quotient Rule with $h(x) = \frac{f(x)}{g(x)}$ and $f(x) = e^x$ and $g(x) = \sin x$. We need to compute the derivative of both the numerator and the denominator: $f'(x) = e^x$ and $g'(x) = \cos x$. Using our Quotient Rule formula, we have that

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} = \frac{(e^x)(\sin x) - (e^x)(\cos x)}{(\sin x)^2} = \frac{e^x(\sin x - \cos x)}{\sin^2 x}.$$